JIGNASA STUDENT STUDY PROJECT-2019

Mathematical Modeling Using Ordinary Differential Equations

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CERTIFICATE

This is to certify that the Project Report entitled "***Mathematical Modeling Using Ordinary Differential Equations",*** submitted to the Commissioner of Collegiate Education Hyderabad, for the best student Project award in **JIGNASA** Competition, was carried out by the following students under my guidance.

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1. Introduction

In science, we explore and understand our real world by observations, collecting data, finding rules inside or among them, and eventually, we want to explore the truth behind and to apply it to predict the future. This is how we build up our scientific knowledge. The above rules are usually in terms of mathematics. They are called mathematical models. A mathematical model is a description of a system using mathematical concepts and language. The process of developing a mathematical model is termed mathematical modeling. Though equations and graphs are the most common types of mathematical models, there are other types that fall into this category. A mathematical model usually describes a system by a set of variables and a set of equations that establish relationships between the variables. One important such models is the ordinary differential equations. It describes relations between variables and their derivatives. Such models appear everywhere. For instance, population dynamics in ecology and biology, mechanics of particles in physics, chemical reaction in chemistry, economics, etc. It is therefore important to learn the theory of ordinary differential equation, an important tool for mathematical modeling and a basic language of science. In this study we will discuss mathematical models using ordinary differential equations

Differential equation: An equation that consists of derivatives is called a differential equation. Differential equations have applications in all areas of science and engineering. Mathematical formulation of most of the physical and engineering problems lead to differential equations.

Differential equations are of two types

- Ordinary differential equation (ODE).
- Partial differential equations (PDE).

Ordinary differential equation: An ordinary differential equation is that in which all the derivatives are with respect to a single independent variable.

Examples

1. $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$, $\frac{dy}{dx}(0)=2, y(0)=4$,

2. $\frac{d^3y}{dx^3}+3\frac{d^2y}{dx^2}+5\frac{dy}{dx}+y=\sin x$, $\frac{d^2y}{dx^2}(0)=12$, $\frac{dy}{dx}(0)=2$, $y(0)=4$

General solution: contains arbitrary constants, e.g.,

$$\frac{d^2h}{dt^2} = g$$

$\Rightarrow \quad h(t) = \frac{1}{2} g\, t^2 + c_1 t + c_2$ where c_1 and c_2 are arbitrary constants .

Particular solution: no arbitrary constants

$$\frac{d^2h}{dt^2} = g \quad \text{with} \quad \frac{dh}{dt}(t=0) = 0, \; h(t=0) = 0 \text{ (initial conditions)}$$

$$\Rightarrow \quad h(t) = \frac{1}{2} g\, t^2$$

Trivial solution: If $y = 0$ is a solution to a differential equation on an interval I, then $y = 0$ is called the trivial solution to that differential equation on I.

e.g.,

$$\frac{dy}{dx} = 3y$$

$y = 0$ (trivial solution)

$y = c\, e^{3x}$ (general solution)

Explicit solution: $y = f(x)$

e.g., $y = c\, e^{3x}$ is an explicit solution of $y' = 3y$

Implicit solution: $f(x, y) = 0$

e.g., $x^2 + y^2 - 1 = 0$ is an implicit solution of $y\, y' = -x$.

Singular solution: A solution can't be obtained from the general solution

e.g., $y'^2 - x\, y' + y = 0$

$y = c\, x - c^2$ general solution

$y = x^2/4$ singular solution

Initial Value Problems

Ordinary Differential Equation + Initial Condition(s)

$$y' = f(x, y) \qquad y(x_0) = y_0$$

Example: $(x^2 + 1)\, y' + (y^2 + 1) = 0, \quad y(0) = 1$

Solution: $$\frac{dy}{y^2 + 1} = -\frac{dx}{x^2 + 1}$$

$$\tan^{-1} y = -\tan^{-1} x + c$$

or $$\tan^{-1} x + \tan^{-1} y = c$$

$$\tan(\tan^{-1} x + \tan^{-1} y) = \tan c$$

or $$\frac{x + y}{1 - xy} = \tan c = c$$ **(general solution)**

Since $y(0) = 1$, we have $\tan c = 1$

$$\Rightarrow \quad \frac{x + y}{1 - xy} = 1$$

$$y = \frac{1 - x}{1 + x}$$ **(particular solution)**

Note that in general there is no arbitrary constant for initial value problems.

2. Objectives

The goal of this study is to learn

(i) How to do mathematical modeling,

(ii) How to solve the corresponding differential equations,

(iii) How to find the solutions

3. Methodology

The need to develop a mathematical model begins with specific questions that the solution of a mathematical model will answer. We use the five basic steps to solve any applied math or application problem. To answer specific questions in a particular application area we wish develop and solve a mathematical "find" problem which in this study will usually be an IVP that is well-posed in a set theoretic sense (i.e., has exactly one solution).

Step 1: UNDERSTAND THE CONCEPTS IN THE APPLICATION AREA.

In order to answer specific questions, we wish to develop a mathematical model (or problem) whose solution will answer the specific questions of interest. Before we can construct a mathematical model, we must first understand the concepts needed from the application area where answers to specific questions are desired. Solution of the model should provide answers to these questions. We start with a description of the phenomenon to be modeled, including the “laws” it must follow (e.g., that are imposed by nature, by an entrepreneurial environment or by the modeler). Recall that the need to answer questions about a ball being thrown up drove us to Newton’s second law, $F = MA$.

Step 2. UNDERSTAND THE MATHEMATICAL CONCEPTS NEEDED.

In order to develop and solve a mathematical model, we must first be sure we know the appropriate mathematics. For this course, we should have previously become reasonably proficient in high school algebra including how to solve algebraic equations and calculus including how to compute derivatives and antiderivatives. We are developing the required techniques and understanding of differential equations. Most of our models will be initial value problems. All of these must be mastered in order to understand the development and solution of mathematical models in science and engineering.

Step 3. DEVELOP THE MATHEMATICAL MODEL.

The model must include those aspects of the application so that its solution will provide answers to the questions of interest. However, inclusion of too much complexity may make the model unsolvable and useless. To develop the mathematical model we use laws that must be followed, diagrams we have drawn to understand the process and notation and nomenclature we developed. Investigation of these laws results in a mathematical model. In this study our models are Initial Value Problems (IVP’s) for a first order ODE that is a rate equation (dynamical system).

In this study we solve the mathematical model which arises from the physical or real world problems using the following methods

- **Separation of Variables**

If the differential equation can be reduced to the form $y' = \frac{f(x)}{g(y)}$

$$g(y)\, y' = f(x)$$

$$y' = \frac{dy}{dx}$$

$$g(y)\, dy = f(x)\, dx$$

then we have a *separable* equation and the general solution can be obtained by integration on both sides

$$\int g(y)\, dy = \int f(x)\, dx + c$$

where c is an arbitrary constant.

Example: $\frac{dy}{dx} = x\sqrt{1 - y^2}$

Solution:

$$\frac{dy}{\sqrt{1 - y^2}} = x\, dx$$

$$\int \frac{dy}{\sqrt{1 - y^2}} = \int x dx + c$$

$$\sin^{-1} y = \frac{x^2}{2} + c$$

$$y = \sin\left(\frac{x^2}{2} + c\right)$$

- **Linear Differential Equations**

An n^{th} order differential equation is *linear* if it can be written in the form

$$\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_o(x)\, y = f(x)$$

Hence, a first-order linear equation has the form

$$\frac{dy}{dx} + p(x)\, y = r(x)$$

e.g., $y' - y = e^{2x}$ 1^{st}– order linear

$y' - \frac{y}{x} = -\frac{5}{2}x^2y^3$ 1^{st}– order nonlinear

Homogeneous Differential Equations

If the function f(x) = 0 [or r(x) = 0], then the above linear differential equation is said to be *homogeneous*; otherwise, it is said to be *nonhomogeneous*.

e.g. $y' - y = 0$ homogeneous

$y' - y = e^{2x}$ nonhomogeneous

Solution of the First-Order Linear Differential Equations

Homogeneous Equation

The solution of the linear homogeneous equation

$$y' + p(x)\, y = 0$$

can be obtained by *separation of variables*

$$\frac{dy}{y} = -\,p(x)\, dx$$

or $$y(x) = c\, e^{-\int p(x)\, dx}$$

Nonhomogeneous Equations

The nonhomogeneous equation $y' + p(x)\, y = r(x)$

can be written in the following form

$$[p(x)\, y - r(x)]\, dx + dy = 0$$

which is of the form

$$P(x)\, dx + Q(x)\, dy = 0$$

with $P(x) = p(x)\, y - r(x)$

$Q(x) = 1$

Since $$\frac{\partial P(x)}{\partial y} = p(x) \neq 0 = \frac{\partial Q}{\partial x}$$

the above equation is not exact differential. However,

$$\frac{\left[\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right]}{Q} = p(x)$$

we have the integrating factor

$$F(x) = e^{\int P(x)dx}$$

for the differential equation. Multiply the differential equation by the integrating factor, we have

$$[y' + p(x)\,y]\,e^{\int p(x)\,dx} = y'e^{\int p(x)dx} + yp(x)e^{\int p(x)dx} = r(x)\,e^{\int p(x)\,dx}$$

According to chain rule, the left side of the above equation is the derivative of

$ye^{\int p(x)\,dx}$, we have

$$\frac{d}{dx}\left[\,y\,e^{\int p(x)\,dx}\right] = r(x)\,e^{\int p(x)\,dx}$$

Integrating both sides of the above equation with respect to x, we have

$$y\,e^{\int p(x)\,dx} = \int r(x)\,e^{\int p(x)\,dx}\,dx) + c$$

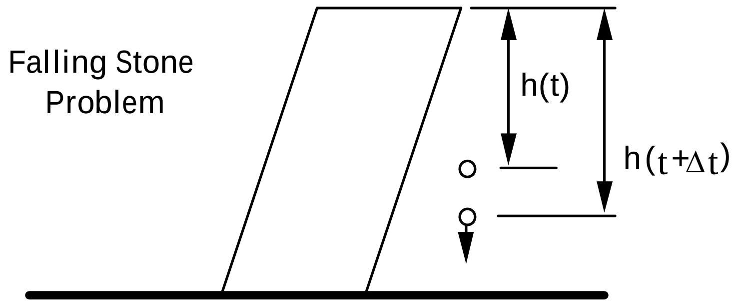
$$y = e^{-\int p(x)\,dx}\left[\int r(x)\,e^{\int p(x)\,dx}\,dx + c\right]$$

No we will discuss some mathematical models which arise in different areas and find the solutions using the above methods.

4. Mathematical models

I. Mathematical model-free falling object.

- Physical Problem
- Mathematical Modeling
- Solving the Math Problem
- Interpretation of its Physical Meaning



F = ma = mg where a = g = acceleration of gravity

Velocity $v(t) = \frac{h(t+\Delta t)-h(t)}{\Delta t}$ for $\Delta t \to 0$, $\mathbf{\frac{dh}{dt} = v}$

Since acceleration $\mathbf{a} = \frac{dv}{dt} = \frac{d^2h}{dt^2}$

we have $\mathbf{\frac{d^2h}{dt^2} = a = g} \approx$ constant

The above equation can be written as

$$\frac{d}{dt}\left[\frac{dh}{dt}\right] = g$$

Mathematical modeling:

$$\frac{d}{dt}\left[\frac{dh}{dt}\right] = g$$

Integrating on both sides, we get

$$\frac{dh}{dt} = g\,t + c_1$$

$$v = g t + c_1$$

and $$h = \frac{1}{2} g t^2 + c_1 t + c_2$$

I.C. (Initial Condition):

$$h(0) = 0 \quad \Rightarrow \quad c_2 = 0$$

$$v(0) = 0 \quad \Rightarrow \quad c_1 = 0$$

Solution of the problem: $h = \frac{1}{2} g t^2$

Interpretation of Physical Meaning

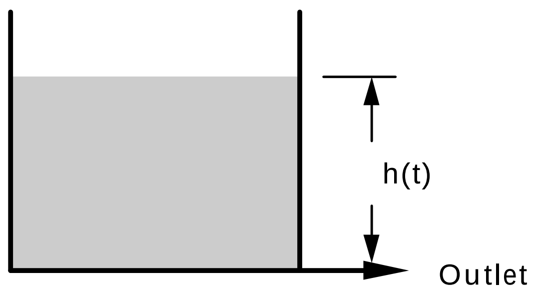
h = falling distance

g = acceleration of gravity

t = time

II. Mathematical model-Torricelli's Law

Time required for draining a tank



$$\Delta V = A\, v\, \Delta t$$

where ΔV = volume of water flows out during Δt

A = cross-sectional area of the outlet = 0.7854 cm^2

v = velocity of the out-flowing water

Torricelli's law states that

$$v = 0.6\sqrt{2gh}$$

where g = acceleration of gravity = 980 cm/sec^2

h = height of water above the outlet

Note that ΔV must equal to change of the volumes of water in the tank, i.e.,

$$\Delta V = -B\ \Delta h \qquad \text{(Mass Balance)}$$

where B = cross-sectional area of the tank = 7854 cm^2

Δh = decrease of the height h(y) of the water

i.e., $A\, v\, \Delta t = -B\, \Delta h$

or $\frac{\Delta h}{\Delta t} = \frac{-Av}{B} = \frac{A\ 0.6\sqrt{2gh}}{B}$

Letting $\Delta t \to 0$, we obtain the differential equation

$$\frac{dh}{dt} = \frac{A\ 0.6\sqrt{2gh}}{B} = -0.00266\sqrt{h}$$

Initially, the height of the water is 150 cm, i.e.,

Initial Condition $h(0) = 150$ cm

Then we have $h(t) = (12.25-0.00133\, t)^2$

III. Mathematical model-Newton's law of Cooling

The rate at which the temperature T(t) changes in a cooling body is proportional to the difference between the temperature of the body and the constant temperature $T(s)$ of the surrounding medium. Symbolically, the rate of change is the derivative and the statement is expressed as

$$\frac{dT}{dt} \propto \left(T - T(\text{s})\right)$$

$$\frac{dT}{dt} = k\left(T - T(s)\right)$$

$$\frac{dT}{\left(T - T(s)\right)} = kdt$$

Integrating on both sides, we get

$$\frac{dT}{dt} \propto \left(T - T(\mathrm{s})\right)$$

$$T - T(s) = e^{kt+c}$$

$$T - T(s) = e^{kt}.e^{c} \Rightarrow T = T(s) + Ae^{kt}$$

$$T \ = \ T(s) + Ae^{kt}$$

Example: Mathematics in Forensic science

A detective discovers a murder victim in a hotel room at 9:00 am one morning. The temperature of the body is 80.0^{o}F. One hour later, at 10:00am, the body has cooled to 75.0^{o} F. The room is kept at a constant temperature of 70.0^{o} F. Assume that the victim had a normal temperature of 98.6^{o} F at the time of death. We will use differential equation to find the time the murder took place.

Let $T(t)$ be the temperature of the body after t hours. By Newton's Law of Cooling we have the differential equation

$$\frac{dT}{dt} = k\left(T - 70\right)$$

where k is a constant . We will solve the differential equation and get a formula for T .

Step.1 $\frac{dT}{\left(T - 70\right)} = kdt$

Step.2 Integrating on both sides

$\ln(T-70) = kt + c$

We may assume $T - 70 \geq 0$ since the body will never be cooler than the room.

Step.3 Solve for T as a function of t, the function will involve the constants k and c.

$(T-70) = e^{kt+c} = e^c e^{kt} = Ae^{kt}$ where $A = e^c$

Then $T = 70 + Ae^{kt}$ ____ (1)

Step.4 Take $t = 0$ when the body was found at 9:00am. Put $t = 0$ and $T = 80.0^o F$ in equation (1) . Then

$80 = 70 + Ae^{k.0} = 70 + A$

$A = 10$ and $T = 70 + 10e^{kt}$

Step.5 After 1 hour, the body temperature is 75.0^o F .

Put $t = 1$ and $T = 75.0^o F$ and solve for k .

$75 = 70 + 10e^{k.1}$

$\frac{1}{2} = e^k$

$k = \ln\left(\frac{1}{2}\right) = -\ln 2$

Thus $T = 70 + 10e^{-t\ln 2}$

Step.6 At what time did the murder take place?

Put $T = 98.6^0 F$ and solve for t.

$$98.6 = 70 + 10e^{-t\ln 2}$$

$$2.86 = e^{-t\ln 2}$$

$$t = -\frac{\ln 2.86}{\ln 2} \approx -1.516$$

1.516 hours is about 1 hour and 31 minutes.

The murder took place at 9:00 A.M - 1 hour and 31 minutes = 7:29am.

IV. Mathematical model- Mixture of Two Salt Solutions

A tank is initially filled with 100 gal of salt solution containing 0.5 lb of salt per gallon. Fresh brine containing 3 lb of salt per gallon runs into the tank at the rate of 2 gal/min, and the mixture, assumed to be kept uniform by stirring, runs out at the same rate. Find the amount of salt in the tank at any time t.

Let Q lb be the total amount of salt in solution in the tank at any time t, and let dQ be the increase in this amount during the infinitesimal interval of time dt. At any time t, the amount of salt per gallon of solution is therefore Q/100 (lb/gal). The material balance of salt in the tank is

$$\left\{\begin{matrix}\text{Rate of Accum.}\\ \text{of Salt}\\ \text{in the Tank}\end{matrix}\right\} = \left\{\begin{matrix}\text{Rate of Salt}\\ \text{Flow}\\ \text{into the Tank}\end{matrix}\right\} - \left\{\begin{matrix}\text{Rate of Salt}\\ \text{Flow}\\ \text{out of the Tank}\end{matrix}\right\}$$

The rate at which salt enters the tank is 2 gal/min × 3 lb/gal = 6 lb/min

Likewise, since the concentration of slat in the mixture as it leaves the tank is the same, as the concentration Q/100 in the tank itself, the rate of salt leaves the tank is

$$2 \text{ gal/min} \times \frac{Q}{100} \text{ lb/gal} = \frac{Q}{50} \text{ lb/min}$$

Hence, the rate of accumulation of salt in the tank dQ/dt is

$$\frac{dQ}{dt} = 6 - \frac{Q}{50}$$

This equation can be written in the form

$$\frac{dQ}{300 - Q} = \frac{dt}{50}$$

and solved as a **separable** equation, or it can be written

$$\frac{dQ}{dt} + \frac{Q}{50} = 6$$

and treated as ***a linear equation*.**

Considering it as a linear equation, we must first compute the integrating factor

$$e^{\int p(x)\,dt} = e^{\int \frac{1}{50} dt} = e^{t/50}$$

Multiplying the differential equation by this factor gives

$$e^{t/50}\left[\frac{dQ}{dt} + \frac{Q}{50}\right] = 6\,e^{t/50}$$

From this, by integration, we obtain

$$Q\,e^{t/50} = 300\,e^{t/50} + c$$

$$Q = 300 + c\,e^{-t/50}$$

Substituting the initial conditions t = 0, Q = 50, we find c = -250

Hence, $Q = 300 - 250\,e^{-t/50}$

V. Mathematical model-Population Growth

The rate of growth of population is proportional to size of the population at time *t*

$$\frac{dN(\mathrm{t})}{dt} \alpha N(\mathrm{t})$$

$$\frac{dN(\mathrm{t})}{dt} = kN(\mathrm{t})$$

where *N(t)* denotes population at time *t* and *k* is a constant of proportionality.

$$\frac{dN(\mathrm{t})}{N(\mathrm{t})} = kdt$$

Integrating on both sides

$$\int \frac{dN(\mathrm{t})}{N(\mathrm{t})} = \int kdt$$

We get $ln\ N(t) = kt + ln C$

$$\ln \frac{N(t)}{C} = kt$$

$N(t) = Ce^{kt}$ can be determined if *N(t)* is given at certain time.

Example: The population of a community is known to increase at a rate proportional to the number of people present at a time t. If the population has doubled in 6 years, how long it will take to triple?

Solution : Let *N(t)* denote the population at time *t*. Let *N(0)* denote the initial population (population at t=0).

$$\frac{dN(\mathrm{t})}{N(\mathrm{t})} = kdt$$

$N(t) = Ce^{kt}$ at $t = 0$ we have $C = N(0)$

Given that the population has doubled in 6 years

$$N(6) = Ce^{6k}$$

But $N(6) = Ce^{6k}$

$$e^{6k} = 2$$

$$k = \frac{1}{6} ln2$$

Our question is how long it will take to triple?

$$N(t) = 3N(0) = 3C$$

$$N(0)\ e^{kt} = 3N(0)$$

$$3 = e^{\frac{1}{6}(\ln 2)t}$$

$$\ln 3 = \frac{(\ln 2)t}{6}$$

$$t = \frac{6\ln 3}{\ln 2} \simeq 9.6 \text{ years}$$

VI. Mathematical model-Radio-active Decay

A radioactive substance decomposes at a rate proportional to its mass. This rate is called the decay rate. If m(t) represents the mass of a substance at any time, then the decay rate $\frac{dm}{dt}$ is proportional to m(t). Let us recall that the half-life of a substance is the amount of time for it to decay to one-half of its initial mass.

Example: A radioactive isotope has an initial mass 200mg, which two years later is 50mg. Find the expression for the amount of the isotope remaining at any time. What is its half-life?

Solution: Let m be the mass of the isotope remaining after t years, and let -k be the constant of proportionality. Then the rate of decomposition is modeled by

$$\frac{dm}{dt} = -kt$$

where minus sign indicates that the mass is decreasing. It is a separable equation. Separating the variables, integrating, and adding a constant in the form lnc, we get

lnm+lnc = $-\ kt$

Simplifying, ln(mc) = $-\ kt$ ______________________________________(1)

$mc = e^{-kt}$

$m = c_1 e^{-kt}$, where $c_1 = \frac{1}{c}$

To find c_1, recall that m =200 when t=0.

Putting these values of m and t in (1) we get

$200 = c_1 \, e^{-ko} = c_1.1$

$c_1 = 200$

and $\quad m = 200e^{-kt}$ ____ (2)

The value of k may now be determined from (2) by substituting t=2, m=150.

$150 = 200 \, e^{-2k}$

or $e^{-2k} = \frac{3}{4}$

or $-2k = \ln \frac{3}{4}$

This gives

$k = \frac{1}{2}\ln\frac{4}{3} = \frac{1}{2}(0.2877) = 0.1438 \approx 0.14$

The mass of the isotope remaining after t years is then given by

$m(t) = 200e^{-.1438t}$

The half-life t_h is the time corresponding to m=100mg.

Thus $100 = 200 \, e^{-0.14t_h}$

or $\frac{1}{2} = e^{-0.14t_h}$

or $t_h = -\frac{1}{0.14}\ln 0.5 = \frac{-0.693}{-0.14} = 4.95$ years

5. Findings

We made an attempt to discuss the modeling phenomena of real world problems and solved them using differential equations. Some of the models included are Newton’s cooling, population growth and decay, free falling of object etc and also studied the application of differential equation in physics and Forensic science. From this study we get some idea how the real world and physical problems are converted into mathematical models and how to solve these models by differential equations.

As a conclusion many fundamental problems in biological, physical sciences and engineering are described by differential equations. It is believed that many unsolved problems of future technologies will be solved using differential equations. On the other hand, physical problems motivate the development of applied mathematics, and this is especially true for differential equations that help to solve real world problems in the field. Thus, making the study on applications of differential equations and their solutions essential in solving the mathematical models with this regard.

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