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A Fixed Point Result with (CLR) Property in S-Metric Spaces

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Abstract

Objective : The present paper is an attempt to improve results on fixed point theorems for four pairwise occasionally weakly compatible (owc) mappings in S-metric spaces. **Method:** We have applied quadratic inequality to prove certain fixed-point results for four pairwise owc mappings under weaker conditions using (CLR) property. **Findings:** We have generalized and expanded some already existing results in the literature and new results are obtained that generated the common fixed points in S-metric spaces. Befitting examples are given to support our findings. **Novelty:** Existence and uniqueness of fixed points in S-metric spaces are established by using (CLR$_{FG}$) property even in the absence of containment conditions.

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1 Introduction

Several authors have introduced various conditions, known as compatible conditions in order to establish the presence of common fixed points. If the two mappings commute (G.Jungck, [1]), it is the simplest technique to find common fixed points. However, because this condition is the strongest one, it is quite common to look for weaker conditions. In 1986, G.Jungck[2] proposed the property of compatibility between two mappings. After that, the idea of weak compatibility was first coined by Jungck and Rhoades[3]. Thagafi and Shahzad[4] presented occasional weak compatibility (owc) between two mappings in 2008, which is a weaker condition than weak compatibility. Aamri and Moutawakil[5] proposed the idea of property (E.A), which is widely used by authors to verify common fixed points. In 2011, Sintunavarat and Kumam[6] introduced a new property, known as (CLR) property that does not demand the closedness of the range of the underlying mappings for the existence of fixed points. Recently, some authors employed this concept to obtain some new fixed point results in various metric spaces[7–12].

The more generalized form of metric space named as S-metric space, was first proposed by S.Sedghi, N.Shobe, A.Aliouche[13] in 2012 as a generalization of G-metric (Z.Mustafa and B.Sims,[14]) and D*-metric (S.Sedghi, N.Shobe and H.Zhou[15]). Many results which were proved earlier in metric spaces are valid in the framework of S-metric spaces, which has generated interest among several researchers. Researchers worked in different directions and a number of remarkable results about the presence of common fixed points in S-metric spaces were obtained [16,17].

2 Methodology

In this section, we present some definitions, examples and lemmas which are required in proving our main results.

Definition 2.1. [13] A function $S : X \times X \times X \to [0,\infty)$, where X is a nonempty set is said to be an S-metric, if for each $u,v,w,a \in X$,

(1) $S(u,v,w) = 0$if and only if $u = v = w$,

(2) $S(u,v,w) \leq S(u,u,a) + S(v,v,a) + S(w,w,a)$

In this case, the pair $(X,\ S)$ is called an S-metric space

Example 2.2. [13] Let $\|.\|$ be a norm on $X = R^n$, then $S(u,v,w) = \|v+w-2u\| + \|v-w\|$for all $u,v,w \in X$ is S-metric on X.

Example 2.3. [18] The function $S : R^3 \to [0,\infty)$defined by $S(u,v,w) = |u-w| + |v-w|$ for all $u,v,w \in R$is an S-metric on R

Lemma 2.4. [13] If X is an S-metric space, then for every

$u,v \in X, S(u,u,v) = S(v,v,u)$

Lemma 2.5. [13] If $\{u_n\}$ and $\{v_n\}$are two sequences in an S-metric space X such that $u_n \to a$ and $v_n \to b$ then $S(u_n,u_n,v_n) \to S(a,a,b)$.

Definition 2.6. [13] A sequence $\{u_n\}$ in an S-metric space X is said to

(i) converge to some $a \in X,\ if\ S(u_n,u_n,a) \to 0$ as $n \to \infty$. We write $\lim_{n\to\infty} u_n = a$

(ii) be a Cauchy sequence if $S(u_n,u_n,u_m) \to 0$ as $m,n \to \infty$

If every Cauchy sequence is convergent in an S-metric space X, then it is said to be complete.

Definition 2.7. Let F and G be two self maps of a set X. Then

(i) a point $u \in X$is said to be a coincidence point of F and G if Fu = Gu.

We denote, $C(F,G) = \{u \in X : Fu = Gu\}$

(ii) the pair (F, G) is called weakly compatible [3] if $FGu = GFu$ for every $u \in X$ such that $Fu = Gu$.

(iii) the pair (F, G) is called occasionally weakly compatible (owc) [4] , if $FGu = GFu$ for some $u \in X$such that $Fu = Gu$.

We define property (E.A) and (CLR) properties in the framework of S-metric spaces as follows.

Definition 2.8. Let F and G be two self maps of an S-metric space X. Then the pair (F, G) is said to satisfy

(i) property (E.A) [5], if there exists a sequence $\{u_n\}$ in X such that $\lim_{n\to\infty} Fu_n = \lim_{n\to\infty} Gu_n = p, p \in X$

(ii) common limit in the range of $G(CLR_G)$property [6] if there exists a sequence $\{u_n\}$ in X such that $\lim_{n\to\infty} Fu_n = \lim_{n\to\infty} Gu_n = p\ where\ p\ \varepsilon\ G\,(X)$.

Example 2.9. Let X = R and consider the S-metric given in Example 2.3. Let the mappings F and G on X be given by $F(u) = 1-u^2$ and $G(u) = u+1$.

For the sequence $\{u_n\}$ given by $u_n = \frac{1}{n^2}, n = 1,2\ldots\ldots$

$$S(Fu_n,Fu_n,1) = S\left(1-\frac{1}{n^4}, 1-\frac{1}{n^4}, 1\right) = \frac{2}{n^4} \to 0, \text{ as } n \to \infty.$$

$$S(Gu_n,Gu_n,1) = S\left(1+\frac{1}{n^2}, 1+\frac{1}{n^2}, 1\right) = \frac{2}{n^2} \to 0, \text{ as } n \to \infty.$$

Therefore $\lim_{n\to\infty} Fu_n = \lim_{n\to\infty} Gu_n = 1 = G(0)$. .

So the pair (F, G) satisfies both (E, A) and (CLR_G) properties.

Definition 2.10. [19] Let M,N,F and G be self maps of an S-metric space X. Then the pairs (M,F) and (N,G) are said to satisfy common limit in the range property with respect to F and G (briefly, (CLR_{FG}) property), if there exists sequences $\{u_n\}$ and $\{v_n\}$in X such that

$\lim_{n\to\infty} Mu_n = \lim_{n\to\infty} Fu_n = \lim_{n\to\infty} Nv_n = \lim_{n\to\infty} Gv_n = p$ where $p \in F(X) \cup G(X)$.

Example 2.11. Let X = R, and consider the S-metric given in Example 2.3. Let the mappings M,N, F and G on X be given by $M(u) = 1-u^2, F(u) = u+1, N(u) = \frac{1}{2}(u+1)\ and\ G(u) = e^{u-1}$

For the sequence $\{u_n\}$ and $\{v_n\}$ given by $u_n = \frac{1}{n^2}$ and $v_n = 1+\frac{1}{n}, n = 1,2,\ldots..$

$S(Mu_n, Mu_n, 1) = S\left(1 - \frac{1}{n^4}, 1 - \frac{1}{n^4}, 1\right) = \frac{2}{n^4} \to 0, \text{ as } n \to \infty$

$S(Fu_n, Fu_n, 1) = S\left(1 + \frac{1}{n^2}, 1 + \frac{1}{n^2}, 1\right) = \frac{2}{n^2} \to 0, \text{ as } n \to \infty$

$S(Nv_n, Nv_n, 1) = S\left(1 + \frac{1}{2n}, 1 + \frac{1}{2n}, 1\right) = \frac{1}{n} \to 0, \text{ as } n \to \infty$

$S(Gv_n, Gv_n, 1) = S\left(e^{\frac{1}{n}}, e^{\frac{1}{n}}, 1\right) = 2\, {}^{\frac{1}{n}} - 1\Big| \to 0, \text{ as } n \to \infty$

Therefore $\lim_{n\to\infty} Mu_n = \lim_{n\to\infty} Fu_n = \lim_{n\to\infty} Nv_n = \lim_{n\to\infty} Gv_n = 1 = F(0) = G(1)$

So the pairs *(M, F)* and *(N, G)* satisfy (CLR_{FG}) property.

Many authors, Tas, Kenan, M. Telci, and B. Fisher [20], Babu and Kameshwari [21], Babu and Alemayehu [22] obtained common fixed points for four maps using quadratic inequality in metric spaces. Babu and Alemayehu [22] used property (E.A) and pair-wise occasional weak compatibility for this purpose. In our present work, we obtain analogous results in S-metric spaces with the same quadratic inequality used in [22]. This study will extend, improve and generalize the results in [22]. We shall give suitable examples to justify our results.

3 Results and Discussion

Proposition 3.1. Let X be an S-metric space and M,N,F and G be four self maps of X satisfying the quadratic inequality

$$\begin{aligned}[S(Mu,Mu,Nv)]^2 \leq c_1 \max\left\{[S(Fu,Fu,Mu)]^2, [S(Gv,Gv,Nv)]^2, [S(Fu,Fu,Gv)]^2\right\} \\ + c_2 \max\{S(Fu,Fu,Mu)S(Fu,Fu,Nv), S(Gv,Gv,Nv)S(Gv,Gv,Mu)\} \\ + c_3 S(Fu,Fu,Nv)S(Gv,Gv,Mu)\end{aligned} \tag{3.1.1}$$

for all $u, v \in X$, where $c_1, c_2, c_3 \geq 0$ and $c_1 + c_3 < 1$

Suppose that either

(i) $M(X) \subseteq G(X)$ and the pair (M,F) satisfies (CLR_F) property; or

(ii) $N(X) \subseteq F(X)$ and the pair (N,G) satisfies (CLR_G) property, holds.

Then $C(M,F) \neq \phi$ and $C(N,G) \neq \phi$

Proof. Suppose that (i) holds.

By the (CLR_F) property of (M, F), it follows that there exists a sequence $\{u_n\}$ in X such that

$$\lim_{n\to\infty} Mu_n = \lim_{n\to\infty} Fu_n = FS, s \in X \tag{3.1.2}$$

$M(X) \subseteq G(X)$ implies that $Mu_n = Gv_n$ for every $n \in N$, for some sequence $\{v_n\}$ in X and hence

$$\lim_{n\to\infty} Gv_n = Fs \tag{3.1.3}$$

Now, we claim that $\lim_{n\to\infty} Nv_n = Fs$

To prove our claim, we take $u = u_n, v = v_n$ in (3.1.1). Then

$$\begin{aligned}[S(Mu_n,Mu_n,Nv_n)]^2 \leq c_1 \max\left\{[S(Fu_n,Fu_n,Mu_n)]^2, [S(Gv_n,Gv_n,Nv_n)]^2, [S(Fu_n,Fu_n,Gv_n)]^2\right\} \\ + c_2 \max\{S(Fu_n,Fu_n,Mu_n)S(Fu_n,Fu_n,Nv_n), S(Gv_n,Gv_n,Nv_n)S(Gv_n,Gv_n,Mu_n)\} \\ + c_3 S(Fu_n,Fu_n,Nv_n)S(Gv_n,Gv_n,Mu_n)\end{aligned}$$

On taking limit superior in the above inequality and using (3.1.2) and (3.1.3),

$\limsup_{n\to\infty} [S(Mu_n,Mu_n,Nv_n)]^2 \leq c_1 \limsup_{n\to\infty} [S(Mu_n,Mu_n,Nv_n)]^2$.

This is a contradiction if $\limsup_{n\to\infty} [S(Mu_n,Mu_n,Nv_n)]^2 \neq 0$, since $c_1 \leq c_1 + c_3 < 1$

So, we must have $\limsup_{n\to\infty} [S(Mu_n,Mu_n,Nv_n)]^2 = 0$,

which implies that $\limsup_{n\to\infty} [S(Mu_n,Mu_n,Nv_n)]^2 = 0$,

Hence,

$$\lim_{n\to\infty} Mu_n = \lim_{n\to\infty} Nv_n = FS. \tag{3.1.4}$$

Now we claim that Ms = Fs.

To prove this, we take $u = s, v = v_n$ in (3.1.1). Then,

$$[S(Ms,Ms,Nv_n)]^2 \leq c_1 \max\left\{[S(Fs,Fs,Ms)]^2, [S(Gv_n,Gv_n,Nv_n)]^2, [S(Fs,Fs,Gv_n)]^2\right\}$$
$$+ c_2 \max\{S(Fs,Fs,Ms)S(Fs,Fs,Nv_n), S(Gv_n,Gv_n,Nv_n)S(Gv_n,Gv_n,Ms)\}$$
$$+ c_3 S(Fs,Fs,Nv_n)S(Gv_n,Gv_n,Ms)$$

On letting $n \to \infty$and using (3.1.3) and (3.1.4), we have $[S(Ms,Ms,Fs)]^2 \leq c_1[S(Ms,Ms,Fs)]^2$
This is a contradiction if $S(Ms,Ms,Fs) \neq 0,$ since $c_1 \leq c_1 + c_3 < 1$
Therefore, we must have

$$Ms = Fs \tag{3.1.5}$$

Hence, $C(M,F) \neq \phi$
Since

$$M(X) \subseteq G(X), Ms = Gt \text{ for some } t \in X \tag{3.1.6}$$

Now, we claim that Nt = Ms.
To prove our claim, we take $u = s, v = t$ in (3.1.1).

Then

$$[S(Ms,Ms,Nt)]^2 \leq c_1 \max\left\{[S(Fs,Fs,Ms)]^2, [S(Gt,Gt,Nt)]^2, [S(Fs,Fs,Gt)]^2\right\}$$
$$+ c_2 \max\{S(Fs,Fs,Ms)S(Fs,Fs,Nt), S(Gt,Gt,Nt)S(Gt,Gt,Ms)\}$$
$$+ c_3 S(Fs,Fs,Nt)S(Gt,Gt,Ms)\}$$

On using (3.1.5) and (3.1.6), we get $[S(Ms,Ms,Nt)]^2 \leq c_1[S(Ms,Ms,Nt)]^2$
This is a contradiction if $S(Ms,Ms,Nt) \neq 0,$ since $c_1 \leq c_1 + c_3 < 1$
Hence, we will have $S(Ms,Ms,Nt) = 0$. This implies Nt = Ms = Gt and hence, $C(N,G) \neq \phi$
In the similar manner, we can prove that C(M,F) is nonempty under assumption (ii).

Theorem 3.2. If the hypothesis of Proposition 3.1 holds and in addition to that, if the pairs (M, F) and (N, G) are occasionally weakly compatible, then the mappings M, N, F and G have a unique common fixed point.

Proof. We can see that $C(M,F) \neq \phi$ and $C(N,G) \neq \phi$ from Proposition 3.1. Since the pair (M,F) is owc $MF_Z = FM_Z$ for some $z \in X$ such that $Mz = Fz = p$. (3.2.1)
$MFz = FMz$ implies $Mp = Fp$
Since the pair (N, G) is owc, $NG_W = GN_W$ for some

$$w \in X \text{ such that } Nw = Gw = q. \tag{3.2.2}$$

$NG_W = GN_W$ implies $N_q = G_q$.
Now let

$$Mp = Fp = p' \text{ and } Nq = Gq = q' \text{ where } p', q' \in X \tag{3.2.3}$$

We now prove that p' = q'.
For this, we consider

$$\left[S\left(p',p',q'\right)\right]^2 = [S(Mp,Mp,Nq)]^2 \leq c_1 \max\left\{[S(Fp,Fp,Mp)]^2, [S(Gq,Gq,Nq)]^2, [S(Fp,Fp,Gq)]^2\right\}$$
$$+ c_2 \max\{S(Fp,Fp,Mp)S(Fp,Fp,Nq), S(Gq,Gq,Nq)S(Gq,Gq,Mp)\}$$
$$+ c_3 S(Fp,Fp,Nq)S(Gq,Gq,Mp)$$

On using (3.2.3), we will have $[S(p',p',q')]^2 \leq (c_1 + c_3)[S(p',p',q')]^2$
which implies

$$p' = q', \text{ since } c_1 + c_3 < 1 \tag{3.2.4}$$

Now we prove that p = q'.
For this, we take

$$\left[S\left(p,p,q'\right)\right]^2 = [S(Mz,Mz,Nq)]^2 \leq c_1 \max\left\{[S(Fz,Fz,Mz)]^2, [S(Gq,Gq,Nq)]^2, [S(Fz,Fz,Gq)]^2\right\}$$
$$+ c_2 \max\{S(Fz,Fz,Mz)S(Fz,Fz,Nq), S(Gq,Gq,Nq)S(Gq,Gq,Mz)\}$$
$$+ c_3 S(Fz,Fz,Nq)S(Gq,Gq,Mz)$$

This implies $[S(p,p,q')]^2 \leq (c_1+c_3)[S(p,p,q')]^2$ on using (3.2.1) and (3.2.3).
Hence,

$$p = q', \text{ as } c_1 + c_3 < 1 \tag{3.2.5}$$

Finally, we prove that p = q.
For this purpose, We take
$[S(p,p,q)]^2 = [S(Mz,Mz,Nw)]^2 \leq c_1 \max\left\{[S(Fz,Fz,Mz)]^2, [S(Gw,Gw,Nw)]^2, [S(Fz,Fz,Gw)]^2\right\}$
$+c_2 \max\{S(Fz,Fz,Mz)S(Fz,Fz,Nw), S(Gw,Gw,Nw)S(Gw,Gw,Mz)\}$
$+c_3 S(Fz,Fz,Nw)S(Gw,Gw,Mz)$
On using (3.2.1) and (3.2.2), we get $[S(p,p,q)]^2 \leq (c_1+c_3)[S(p,p,q)]^2$
which implies that

$$p = q, \text{ since } c_1 + c_3 < 1 \tag{3.2.6}$$

From (3.2.4), (3.2.5) and (3.2.6), we have $p' = q' = p = q$
From (3.2.3), it follows that

$$Mp = Fp = p = Np = Gp \tag{3.2.7}$$

To prove that p is unique, we suppose that p* is a common fixed point of M,N,F and G other than p.
Therefore,

$$Mp^* = Fp^* = p^* = Np^* = Gp^* \tag{3.2.8}$$

Then from the inequality (3.2.1),
$[S(Mp,Mp,Np^*)]^2 \leq c_1 \max\{S(Fp,Fp,Mp)]^2, [S(Gp^*,Gp^*,Np^*)]^2, [S(Fp,Fp,Gp^*)]^2\big\}$
$+c_2 \max\{S(Fp,Fp,Mp)S(Fp,Fp,Np^*), S(Gp^*,Gp^*,Np^*)S(Gp^*,Gp^*,Mp)\}$
$+c_3 S(Fp,Fp,Np^*)S(Gp^*,Gp^*,Mp)$
On using (3.2.7) and (3.2.8), we get $[S(p,p,p^*)]^2 \leq (c_1+c_3)[S(p,p,p^*)]^2$
which is a contradiction, since $c_1+c_3<1$
Therefore, we must have p = p*.
Hence Theorem 3.2 follows.
Example 3.3. Let X = *[0,1]* and consider the S - metric given in Example 2.3.
Then the inequality (3.1.1) will be

$$\begin{aligned} |Mu-Nv|^2 &\leq c_1 \max\left\{|Fu-Mu|^2, |Gv-Nv|^2, |Fu-Gv|^2\right\} \\ &+c_2 \max\{|Fu-Mu||Fu-Nv|, |Gv-Nv|||Gv-Mu|\} + c_3|Fu-Nv||Gv-Mu| \end{aligned} \tag{3.3.1}$$

Let the mappings M, N, F and G on X be defined by
$M(u) = \begin{cases} 0 & \text{if } u \in \left[0, \frac{4}{5}\right) \\ \frac{1}{10} & \text{if } u \in \left[\frac{4}{5}, 1\right] \end{cases} \qquad N(u) = 0, \forall u \in [0,1]$
$F(u) = \begin{cases} u & \text{if } u \in \left[0, \frac{4}{5}\right) \\ \frac{9}{10} & \text{if } u \in \left[\frac{4}{5}, 1\right] \end{cases} \qquad G(u) = \begin{cases} \frac{u}{20} & \text{if } u \in \left[0, \frac{4}{5}\right) \\ \frac{4}{5} & \text{if } u \in \left[\frac{4}{5}, 1\right] \end{cases}$
Here we observe that neither *F(X)* nor *G(X)* are closed.
We can also observe that $N(X) \subseteq F(X)$, but $M(X) \not\subset G(X)$.
Case I: For $u \in \left[0, \frac{4}{5}\right)$ and for every $v \in [0,1], Mu - Nv = 0$
Therefore, $|Mu - Nv| = 0$
Hence inequality (3.3.1) holds.
Case II: For $u \in \left[\frac{4}{5}, 1\right]$ and for every $v \in [0,1], \quad Mu = \frac{1}{10}, Fu = \frac{9}{10}$ and $Nv = 0$
Therefore, $|Mu - Nv| = \frac{1}{10}, |Fu - Mu| = \frac{8}{10}$

$$|Mu-Nv|^2 = \frac{1}{100} < \frac{8}{25} = \frac{1}{2}|Fu-Mu|^2 \le \frac{1}{2}\max\left\{|Fu-Mu|^2, |Gv-Nv|^2, |Fu-Gv|^2\right\}$$

Hence,

$$+c_2 \max\{|Fu-Mu||Fu-Nv|, |Gv-Nv||Gv-Mu|\}$$
$$+c_2|Fu-Nv||Gv-Mu|$$

Then the inequality (3.3.1) holds for $c_1 = \frac{1}{2}, c_3 = \frac{1}{3}$ and $c_2 \ge 0$

Thus, the inequality (3.3.1) holds in both the cases for $c_1 = \frac{1}{2}, c_3 = \frac{1}{3}$ and $c_2 \ge 0$, where $c_1 + c_3 = \frac{5}{6} < 1$

Also for the sequence $\{u_n\}$ in X given by $u_n = \frac{1}{n^3}, n = 1,2,3\ldots..$

$S(Nu_n, Nu_n, 0) = 0, S(Gu_n, Gu_n, 0) = S\left(\frac{1}{20n^3}, \frac{1}{20n^3}, 0\right) = \frac{1}{10n^3} \to 0$ as $n \to \infty$.

Thus $\lim_{n\to\infty} Nu_n = \lim_{n\to\infty} Gu_n = 0 = G(0)$

The pair of mappings (N,G) follows (CLR$_G$) property.

Furthermore, (M,F) and (N,G) are occasionally weakly compatible.

We can also see that '0' is the only common fixed point of M,N,F and G.

Remark 3.4. By replacing (CLR$_F$) / (CLR$_G$) property with (CLR$_{FG}$) property, the containment conditions can be removed from the assumptions for proving existence of fixed points, which can be seen in the following proposition and theorem.

Proposition 3.5. Let (X,S) be an S-metric space and M,N,F and G be four self mappings of X satisfying the quadratic inequality

$$\begin{aligned}[S(Mu,Mu,Nv)]^2 \le & c_1 \max\left\{[S(Fu,Fu,Mu)]^2, [S(Gv,Gv,Nv)]^2, [S(Fu,Fu,Gv)]^2\right\} \\ & + c_2 \max\{S(Fu,Fu,Mu)S(Fu,Fu,Nv), S(Gv,Gv,Nv)S(Gv,Gv,Mu)\} \\ & + c_3 S(Fu,Fu,Nv)S(Gv,Gv,Mu).\end{aligned} \tag{3.5.1}$$

for all $u,v \in X$, where $c_1, c_2, c_3 \ge 0$ and $c_1 + c_3 < 1$

Suppose that the pair (M,F) and (N,G) satisfy (CLR$_{FG}$) property.

Then $C(M,F) \ne \phi$ and $C(N,G) \ne \phi$

Proof. Since the pairs (M,F) and (N,G) satisfy (CLR$_{FG}$) property, there exists sequences $\{u_n\}$ and $\{v_n\}$ in X such that

$$\lim_{n\to\infty} Mu_n = \lim_{n\to\infty} Fu_n = \lim_{n\to\infty} Nv_n = \lim_{n\to\infty} Gv_n = p \in F(X) \cap G(X) \tag{3.5.2}$$

$$\text{Therefore } p = Fz = Gw \text{ for some } z, w \in X. \tag{3.5.3}$$

We now prove that Mz = p.

This can be done by putting $u = z$ and $v = v_n$ in (3.5.1). Then

$$\begin{aligned}[S(Mz,Mz,Nv_n)]^2 \le & c_1 \max\left\{[S(Fz,Fz,Mz)]^2, [S(Gv_n,Gv_n,Nv_n)]^2, [S(Fz,Fz,Gv_n)]^2\right\} \\ & + c_2 \max\{(Fz,Fz,Mz)S(Fz,Fz,Nv_n), S(Gv_n,Gv_n,Nv_n)S(Gv_n,Gv_n,Mz)\} \\ & + c_3 S(Fz,Fz,Nv_n)S(Gv_n,Gv_n,Mz)\end{aligned}$$

On letting $n \to \infty$ and using (3.5.2) and (3.5.3), we get $[S(Mz,Mz,p)]^2 \le c_1[S(p,p,Mz)]^2$

On using (3.5.3) and (3.5.4), we get

$$\text{Since } c_1 < 1, \text{ this implies } Mz = p. \tag{3.5.4}$$

From (3.5.3) and (3.5.5), we have Mz = Fz, which implies $C(M,F) \ne \phi$

We now prove that Nw = p.

This can be done by putting $u = z$ and $v = w$ in (3.5.1). Then

$$\begin{aligned}[S(Mz,Mz,Nw)]^2 \le & c_1 \max\left\{[S(Fz,Fz,Mz)]^2, [S(Gw,Gw,Nw)]^2, [S(Fz,Fz,Gw)]^2\right\} \\ & + c_2 \max\{S(Fz,Fz,Mz)S(Fz,Fz,Nw), S(Gw,Gw,Nw)S(Gw,Gw,Mz)\} \\ & + c_3 S(Fz,Fz,Nw)S(Gw,Gw,Mz)\end{aligned}$$

On using (3.5.3) and (3.5.4), we get $[S(p,p,Nw)]^2 \le c_1[S(p,p,Nw)]^2$

$$\text{Since } c_1 < 1, \text{ this implies } Nw = p. \tag{3.5.5}$$

From (3.5.3) and (3.5.5), it follows that $N_w = G_w$. Hence $C(N,G) \ne \phi$

Theorem 3.6. If the hypothesis of Proposition 3.5 holds and in addition to that, if the pairs (M,F) and (N,G) are occasionally weakly compatible, then the mappings M,N,F and G have a unique common fixed point.

Proof. We can see that $C(M,F) \neq \phi$ and $C(N,G) \neq \phi$ from Proposition 3.5.

The rest of the proof is identical to the proof of the theorem (3.2).

Example 3.7. Let X = [0,1] and consider the S-metric given in Example 2.3.

Then the inequality (3.1.1) will be

$$\begin{aligned} |Mu-Nv|^2 \leq & c_1 \max\left\{|Fu-Mu|^2, |Gv-Nv|^2, |Fu-Gv|^2\right\} \\ & + c_2 \max\{|Fu-Mu||Fu-Nv|, |Gv-Nv||Gv-Mu\ |\} \\ & + c_3|Fu-Nv||Gv-Mu|. \end{aligned} \tag{3.7.1}$$

Let the mappings *M,N,F* and *G* on *X* be defined by

$M(u) = \begin{cases} \frac{9}{10} & \text{if } u \in \left[0, \frac{4}{5}\right) \\ \frac{4}{5} & \text{if } u \in \left[\frac{4}{5}, 1\right] \end{cases} \quad N(u) = \begin{cases} 1 & \text{if } u \in \left[0, \frac{4}{5}\right) \\ \frac{4}{5} & \text{if } u \in \left[\frac{4}{5}, 1\right] \end{cases}$

$F(u) = \begin{cases} 0 & \text{if } u \in \left[0, \frac{4}{5}\right) \\ 2-\frac{3}{2}u & \text{if } u \in \left[\frac{4}{5}, 1\right] \end{cases}, \quad G(u) = \begin{cases} \frac{1}{10} & \text{if } u \in \left[0, \frac{4}{5}\right) \\ 1-\frac{1}{4}u & \text{if } u \in \left[\frac{4}{5}, 1\right] \end{cases}$

Here we see that

$N(X) = \left\{1, \frac{4}{5}\right\} \not\subseteq \{0\} \cup \left[\frac{1}{2}, \frac{4}{5}\right] = F(X)$ and $M(X) = \left\{\frac{9}{10}, \frac{4}{5}\right\} \not\subset \left\{\frac{1}{10}\right\} \cup \left[\frac{3}{4}, \frac{4}{5}\right] = G(X)$.

Case I: For $u, v \in \left[0, \frac{4}{5}\right), |Mu-Nv|^2 = \left|\frac{9}{10}-1\right| = \frac{1}{100} < \frac{81}{200} = \frac{1}{2}|Fu-Mu|^2$

Case II: For $u \in \left[0, \frac{4}{5}\right)$ and $v \in \left[\frac{4}{5}, 1\right), |Mu-Nv|^2 = \left|\frac{9}{10}-\frac{4}{5}\right|^2 = \frac{1}{100} < \frac{81}{200} = \frac{1}{2}|Fu-Mu|^2$.

Case III: For $u, v \in \left[\frac{4}{5}, 1\right), \quad |Mu-Nv|^2 = 0 < \frac{1}{2}|Fu-Mu|^2$

Case IV: For $u \in \left[\frac{4}{5}, 1\right), v \in \left[0, \frac{4}{5}\right), |Mu-Nv|^2 = \left|\frac{4}{5}-1\right|^2 = \frac{1}{25} < \frac{81}{200} = \frac{1}{2}|Gv-Nv|^2$

We observe that the inequality (3.7.1) holds in all the cases for $c_1 = \frac{1}{2}, c_3 = \frac{1}{3}$and $c_2 \geq 0$, where $c_1 + c_3 = \frac{5}{6} < 1$

Also for the sequences, $u_n = \frac{4}{5} + \frac{1}{5n^2}$ and $v_n = \frac{4}{5} + \frac{1}{5n}, n = 1, 2 \ldots\ldots..$ in X,

$S\left(Mu_n, Mu_n, \frac{4}{5}\right) = S\left(\frac{4}{5}, \frac{4}{5}, \frac{4}{5}\right) = 0$

$S\left(Fu_n, Fu_n, \frac{4}{5}\right) = S\left(\frac{4}{5} - \frac{3}{10n^2}, \frac{4}{5} - \frac{3}{10n^2}, \frac{4}{5}\right) = \frac{3}{5n^2} \rightarrow 0$ as $n \rightarrow \infty$

$S\left(Nv_n, Nv_n, \frac{4}{5}\right) = S\left(\frac{4}{5}, \frac{4}{5}, \frac{4}{5}\right) = 0$

$S\left(Gv_n, Gv_n, \frac{4}{5}\right) = S\left(\frac{4}{5} - \frac{1}{20n}, \frac{4}{5} - \frac{1}{20n}, \frac{4}{5}\right) = \frac{1}{10n} \rightarrow 0$ as $n \rightarrow \infty$

Thus$\lim_{n\to\infty} Mu_n = \lim_{n\to\infty} Fu_n = \lim_{n\to\infty} Nv_n = \lim_{n\to\infty} Gv_n = \frac{4}{5} = F\left(\frac{4}{5}\right) = G\left(\frac{4}{5}\right)$.

The pairs *(M,F)* and *(N,G)* satisfy (CLR_{FG}) property

Moreover, *(M,F)* and *(N,G)* are occasionally weakly compatible.

We can also see that 4/5 is the only common fixed point of M,N,F and G and it is evidently seen that common fixed points exist also in the absence of containment conditions, if we apply (CLR_{FG}) property.

4 Conclusion

In Theorem 3.2, without assuming the closedness of the range, by applying (CLR) property a common fixed point is established for four pairwise owc maps. In Theorem 3.6, with the help of (CLR$_{FG}$) property alone, without assuming the containment conditions a common fixed point is obtained. Further all the results are supported with suitable examples.

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