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New classes of K-uniformly convex and starlike functions with respect to symmetric points

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ABSTRACT
In this paper, we introduce a new subclass of k-uniformly starlike and convex functions with respect to symmetric points. We provide necessary and sufficient conditions, coefficient estimates, distortion bounds, extreme points and radii of close to convexity, star likeness and convexity for the functions in this class. We also obtain integral transforms for the functions in this class.

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1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions given by $$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

which are analytic in the open unit disc

$U = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$

Let *S* denote the subclass of $\mathcal{A}$ consisting of functions that are univalent in U.

Let S^* the subclasses of *S* consisting of functions starlike in U. It is known that $f \in S^*$ if and only if $\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in U$.

Goodman [2, 3] introduced the concept of uniform starlikeness and uniform convexity for functions in *A*. A function $f(z)$ is said to be uniformly convex if $f(z)$ is convex and has the property that each circular arc γ contained in *U* with center ξ is also in *U*, the arc $f(\gamma)$ is convex. Similarly the function $f(z)$ is uniformly starlike if $f(z)$ is starlike and has the property that for each circular arc γ is contained in *U* with center ξ is also in *U*, the arc $f(\gamma)$ is starlike. The classes of functions consisting of uniformly convex and starlike functions are denoted by *UCV* and *UST* respectively.

The following analytic Characterization of *UCV* and *UST* are obtained by Goodman [2, 3]. The class of uniformly convex functions (*UCV*) consists of functions $f \in A$ satisfying,

$$\mathrm{Re}\left\{1 + (z-\xi)\frac{f''(z)}{f'(z)}\right\} \geq 0, \qquad \forall, z, \xi \in U$$

The class of uniformly starlike functions (*UST*) consists of functions $f \in A$ satisfying

$$\mathrm{Re}\left\{\frac{(z-\xi)f'(z)}{f(z)-f(\xi)}\right\} \geq 0, \ \forall\ z, \xi \in U$$

Ronning.F [8], Ma and Minda [4] have individually given the following one variable characterization for the function *f* in *UCV* and *UST* classes.

A function $f \in A$ is said to be in the class *UCV* if and only if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right|, \quad \forall\, z \in U$$

Let the class of functions $f(z)$ for which there is a uniformly convex function $F(z)$ such that $f(z) = zF'(z)$, be denoted by S_p. It is easy to see that the function $f(z)$ is in S_p if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad \forall\, z \in U$$

Recently many research workers has extended or generalized the classes *UST*, *UCV* and the class *Sp*. Recently S. Shams, S.R. Kulkarni and J.M. Jahangiri [10] introduced the classes *SD* (k, β) and *KD* (k, β) to be the classes of functions $f \in A$ satisfying the conditions

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > k\left|\frac{zf'(z)}{f(z)} - 1\right| + \beta$$

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > k\left|\frac{zf''(z)}{f'(z)}\right| + \beta$$ respectively for some $k \geq 0$ and $\beta\ (0 \leq \beta < 1)$

It is noted that $f(z) \in KD(k, \beta)$ if and only if $zf'(z) \in SD(k, \beta)$. They have shown some sufficient conditions for $f(z)$ to be in the classes $SD\,(k, \beta)$ and $KD\,(k, \beta)$.

By imposing the condition $0 \leq k \leq \beta$, S. Owa.Y, Polatoglu and E. Yavuz [5] obtained, coefficient inequalities, distortion properties for the functions in the classes *SD* (k, β) and *KD* (k, β).

H.M. Srivastava, T.N. Shanmugam, C. Ramachandran and S. Siva Subramanian [12] defined and studied the class $U\,(\lambda, \alpha, \beta, k)$ to be the class of functions $f \in T$ for which

$$\operatorname{Re}\left(\frac{zF'(z)}{F(z)}\right) > k\left|\frac{zF'(z)}{F(z)} - 1\right| + \beta$$

$(0 \leq \alpha \leq \lambda \leq 1)$ $(0 \leq \beta < 1)$ & $k \geq 0$.

Where F(z) = $\lambda\alpha z^2 f''(z) + (\lambda - \alpha)\, zf'(z) + (1 - \lambda + \alpha)\, f(z)$

They have obtained the coefficient inequalities, necessary sufficient conditions, and distortion properties convex linear combinations, radius of starlikeness, convexity and integral operators for the functions in this class.

Sakaguchi [9] defined the class of starlike functions with respect to symmetric points as follows.

A function $f \in S$ is said to be starlike with respect to symmetric points in U iff

$$\operatorname{Re}\left(\frac{2zf'(z)}{f(z) - f(-z)}\right) > 0, \quad z \in U$$

We denote this class by S_s^*.

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Many researchers have discussed the following class of functions convex with respect to symmetric points and its subclasses [6, 7, 11, 13]

A function $f \in S$ is said to be convex with respect to symmetric points in U if and only if

$$\operatorname{Re}\left(\frac{\left[zf'(z)\right]'}{f'(z)+f'(-z)}\right)>0, \qquad \forall z \in U$$

Let T denote the class consisting of functions of the form

$$f(z)=z-\sum_{n=2}^{\infty} a_n z^n \tag{1.2}$$

Here a_n is a non-negative real number.

Silverman [11] Introduced and investigated the following subclasses of T.

$$T^*(x)=S^*(\alpha)\cap T \ , \ C(\alpha)=K(\alpha)\cap T \quad (0 \le \alpha < 1)$$

In this paper, we introduce the class $U_s(\lambda,\alpha,k,\beta)$ of functions regular in U given by (1) and defined as follows.

Definition: A function $f(z) \in A$ is said to be in the class $U_s(\lambda,\alpha,k,\beta)$ if for all $z \in U$.

$$\operatorname{Re}\left\{\frac{2zF'(z)}{F(z)-F(-z)}\right\} > k\left|\frac{2zF'(z)}{F(z)-F(-z)}-1\right|+\beta \tag{1.3}$$

Where $F(z)=\lambda\alpha z^2 f''(z)+(\lambda-\alpha)zf'(z)+(1-\lambda+\alpha)f(z)$

For suitable values $\lambda,\ \alpha,\ k,\ \beta$ of the class of functions $U_s(\lambda,\alpha,k,\beta)$ reduces to various new classes of regular functions.

We also observe that

i. $U_s(0,0,0,0)=s_s^*$ and

ii. $U_s(1,0,0,0)=c_s^*$

We now let $TU_s(\lambda,\alpha,k,\beta)=U_s(\lambda,\alpha,k,\beta)\cap T$.

2. Coefficient inequalities

We employ the technique adopted by Aqlan et.al [1] to find the coefficient estimates for the function class $TU_s(\lambda,\alpha,k,\beta)$.

Theorem 2.1: A function $f \in TU_s(\lambda,\alpha,k,\beta)$ if and only if

$$\sum_{122}^{\infty}\left[\left(2n(1+k)-(k+\beta)\right)\right]\left[(n-1)(n\lambda\alpha+\lambda-\alpha)+1\right]+ \\ (k+\beta)\left(n(n-1)\lambda\alpha-(\lambda-\alpha)(n+1)+1\right)(-1)^n]a_n \\ 2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]$$

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for $0 \le \alpha \le \lambda \le 1,\ 0 \le \beta < 1,\ k \ge 0$.

Proof: Let a factorial $f \in T$ and satisfies the condition (2.1). We will show that $f \in TU_s(\lambda, \alpha, k, \beta)$

Applying the principle

$$\operatorname{Re}(w) > k|w-1| + \beta \Leftrightarrow \operatorname{Re}\left(w\left(1+ke^{i\theta}\right) - ke^{i\theta}\right)_{S\beta} \qquad (2.2)$$
$$\left(-\pi \le \theta \le \pi,\ 0 \le \beta < 1,\ k \ge \theta\right)$$

For the function $w(z) = \dfrac{2zF'(z)}{F(z) - F(-z)}$ on right hat side, we get

$$\operatorname{Re}\left[\frac{2zF'(z)\left(1+ke^{i\theta}\right) - ke^{ia}\left[F(z) - F(-z)\right]}{F(z) - F(-z)}\right] > \beta \qquad (2.3)$$

By setting

$$G(w) = 2ZF'(z)\left(1+ke^{i\theta}\right) - ke^{i\theta}\left[F(Z) - F(-z)\right]$$
$$H(z) = F(z) - F(-z) \qquad ,$$

The above inequality (2.3) is equivalent to

$$\left|G(z) + (1-\beta)H(z)\right| > \left|G(z) - (1+\beta)H(z)\right|$$

Consider

$$\left|G(z) + (1-\beta)H(z)\right|$$
$$\left|(4-2\beta+2\alpha-2\lambda)z - 2k(\alpha-\lambda)i^{i\theta}z - (2\alpha-2\lambda)\beta z\right|$$
$$-\sum_{122}^{\infty}\left[\left[2n\left[(n-1)(n\lambda\alpha+\lambda-\alpha)+1\right] + (1-\beta)\begin{bmatrix}\left[(n-1)(n\lambda\alpha+\lambda+\alpha)+1\right] \\ -\left[n(n-1)\lambda\alpha - (\lambda-\alpha)(n+1)+1\right]\end{bmatrix}(-1)^n\right]\right]a_n z^n$$
$$-\sum_{122}^{\infty}\left[(2n-1)\left[(n-1)(n\lambda\alpha+\lambda-\alpha)+1\right] + \left[n(n-1)\lambda\alpha(\lambda-\alpha)(n+1)+1\right]\right]a_n z^n$$
$$\ge (4-2\beta+2\alpha-2\lambda)|z| - 2k(\alpha-\lambda)|z| - 2\beta(\alpha-\lambda)|z|$$
$$-\sum_{122}^{\infty}\begin{bmatrix}\left[2n+(1-\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda-\alpha)+1\right] - (1-\beta) \\ \left[n(n-1)\lambda\alpha - (\lambda-\alpha)(n+1)+1\right](-1)^n\end{bmatrix} a_n|z|^n - k[$$
$$\sum_{122}^{\infty}\left[(2n-1)\left[(n-1)(n\lambda\alpha+\lambda-\alpha)+1\right] + \left[n(n-1)+\alpha-(\lambda-\alpha)(n+1)+1\right](-1)^n\right] a_n|z|^n$$
$$\ge \left[4-2\beta-(2\alpha-2\lambda)(k+\beta-1)\right]|z| -$$
$$\sum_{n=2}^{\infty}\begin{bmatrix}\left[2n+(1-\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda-\alpha)+1\right] - \\ (1-\beta)\left(n(n-1)\lambda\alpha - (\lambda-\alpha)(n+1)+1\right)\end{bmatrix} a_n|z|^n$$

$$-k\left[\sum_{n=2}^{\infty}(2n-1)\big((n-1)(n\lambda\alpha+\lambda-\alpha)+1\big)\left[n(n-1)\lambda\alpha-(\lambda-\alpha)(n+1)+1\right](-1)^n\right]a_n|z|^n$$

(2.5)

Consider similarly

$$|F(z)-(1\beta)1+(z)|$$

$$=\left|-2\beta z+\left[-(2\alpha-2)\right][k+\beta+1]z+\sum_{n=2}^{\infty}\left[-[2n[]]\right|$$

$$\left[(n-1)(n\lambda\alpha+\lambda-\alpha)+1\right]-(1+\beta)\left[(n-1)(n\lambda\alpha+\lambda-\alpha)+1\right]$$

$$+(1+\beta)\left[n(n-1)\lambda\alpha-(\lambda-\alpha)(n+1)+1\right]a_n z^n\ +$$

$$\sum_{n=2}^{\infty}\left[-\left[(2n-1)\left[(n-1)(n\lambda\alpha+\lambda-\alpha)+1\right]+\left[n(n-1)\lambda\alpha-(\lambda-\alpha)(n+1)+1\right](-1)^{-1}\right]\right]$$

$$\le 2\beta|z|+(2\alpha-2\lambda)(1+k+\beta)|z|\ +$$

$$\sum_{n=2}^{\infty}\begin{bmatrix}2n\left[(n-1)(n\lambda\alpha+\lambda-\alpha)+1\right]-(1+\beta)\left[(n-1)(n\lambda\alpha+\lambda-\alpha)+1\right]+\\(1+\beta)\left[n((n-1)\lambda\alpha)-(\lambda-\alpha)(n+1)+1\right]a_n|z^n|+\end{bmatrix}$$

$$\sum_{n=2}^{\infty}\left[(2n-1)\left[(n-1)(n\lambda\alpha+\lambda-\alpha)+1\right]+\left[n(n-1)\lambda\alpha-(\lambda-\alpha)(n+1)+1\right](-1)^n\right]a_n|z|^n$$

(2.6)

Therefore from inequalities (2.5) and (2.6) we have

$$|G(z)+(1-\beta)H(z)|-|G(z)-(1+\beta)H(z)|$$

$$\ge\left[4(1-\beta)-4(\alpha-\lambda)(k+\beta)\right]-2\sum_{n=2}^{\infty}\begin{bmatrix}\left[\big(2n(k+1)-(k+\beta)\big)(n-1)(n\lambda\alpha+\lambda-\alpha)+1\right]+\\(k+\beta)\big(n(n-1)\lambda\alpha-(n+1)+1\big)(-1)^n\end{bmatrix}a_n|z|^n$$

$$\ge\left[4(1-\beta)-(\alpha-\lambda)(k+\beta)\right]-2\sum_{n=2}^{\infty}\begin{bmatrix}\left[2n(k+1)-(k+\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda-\alpha)+1\right]+\\(k+\beta)\big(n(n-1)\lambda\alpha-(\lambda-\alpha)(n+1)+1\big)(-1)^n\end{bmatrix}a_n|z|^n$$

≥ 0 using the result in (2.1)

$$\Rightarrow \operatorname{Re}\left\{\frac{2zF'(z)}{F(z)-F(-z)}\right\}>k\left|\frac{2zF'(z)}{F(z)-F(-z)}-1\right|+\beta \quad \text{from (2.2)}$$

Hence $f\in TU_s(\lambda,\alpha,k,\beta)$

Conversely suppose $f\in TU_s(\lambda,\alpha,k,\beta)$

By setting $0\le|z|=\gamma<1$ and choosing the values of z on the positive real axis, then from inequality (2.3), we have

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$$\operatorname{Re}\frac{\begin{bmatrix}2\gamma-(2\alpha-2\lambda)\gamma k e^{i\theta}-2\sum n\left[(n-1)(n\lambda\alpha+\lambda-\alpha)+1\right]a_n\gamma^n-\\-\sum_{n=2}^{\infty}\left[(n-1)(n\lambda\alpha+\lambda-\alpha)+1\right][2n-1]ke^{i\theta}a_n z^n\\-\sum\left[n(n-1)\lambda\alpha-(\lambda-\alpha)(n+1)+1\right](-1)^n a_n\gamma^n ke^{i\theta}\end{bmatrix}}{(2\alpha-2\lambda+2)\gamma+\beta\sum_{n=2}^{\infty}\left[\begin{bmatrix}n(n-1)\lambda\alpha+(\lambda-\alpha)(n-1)+1-\\\left[n(n-1)\lambda\alpha-(n-\alpha)(n+1)+1\right](-1)^n\end{bmatrix}a_n\gamma^n\right]}>\beta \quad (2.7)$$

In view of the elementary identity

$$\operatorname{Re}0\left(-e^{i\theta}\right)\geq-\left|e^{i\theta}\right|=-1$$

Then the above inequality (2.7) becomes

$$\operatorname{Re}\frac{\begin{bmatrix}2\left[(\gamma-\beta)-(\alpha-\lambda)(k+\beta)\right]\gamma-\sum_{n=2}^{\infty}\left[2n(1+k)-(k+\beta)\right]\\\left[(n-1)(n\lambda\alpha+\lambda+\alpha)+1\right]a_n\gamma^n-(k+\beta)\left[n(n-1)\lambda\alpha-(\lambda-\alpha)(n+1)+1\right](-1)^n a_n\gamma^n\end{bmatrix}}{(2\alpha-2\lambda+2)\gamma+\sum_{n=2}^{\infty}\left[\begin{bmatrix}n(n-1)\lambda\alpha+(\lambda-\alpha)(n-1)+1-\\\left[n(n-1)\lambda\alpha-(n-\alpha)(n+1)+1\right](-1)^n\end{bmatrix}a_n\gamma^n\right]}\geq 0\cdot$$

(2.8)

Letting in (2.8) we get

$$\sum_{n=2}^{\infty}\begin{bmatrix}\left[2n(1+k)-(k+\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda-\alpha)+1\right]-\\(k+\beta)\left[\left(\lambda(n-1)\lambda\alpha-(\lambda-\alpha)(n+1)+1\right)(-1)^n\right]\end{bmatrix}a_n\leq 2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]$$

This is the result in (2.1)

Hence the theorem.

Theorem 2.2: If $f\in TU_s(\lambda,\alpha,k,\beta)$,

$$a_n\leq\frac{2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]}{\begin{matrix}\left[2n(1+k)-(k+\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda+\alpha)+1\right]-\\\left[(k+\beta)\left[n(n-1)\lambda\alpha\right]-(\lambda-\alpha)(n+1)+1\right](-1)^n\end{matrix}}\ \forall n\geq 2 \qquad (2.9)$$

3. Distortion and covering theorems for the function class $TU_s(\lambda,\alpha,k,\beta)$

Theorem 3.1: If the function $f\in TU_s(\lambda,\alpha,k,\beta)$, then

$$\begin{matrix}r-\frac{2(1-\beta)-(2\alpha-2\lambda)(k+\beta)}{\begin{matrix}\left[4(1+k)-(k+\beta)\right]\left[(2\lambda\alpha+\lambda-\alpha)+1\right]+\\(k+\beta)\left[2\lambda\alpha-3(\lambda-\alpha)+1\right]\end{matrix}}r^2\leq|f(z)|\leq\\r+\frac{2(1-\beta)-(2\alpha-2\lambda)(k+\beta)}{\begin{matrix}\left[4(1+k)-(k+\beta)\right]\left[(2\lambda\alpha+\lambda-\alpha)+1\right]+\\(k+\beta)\left[2\lambda\alpha-3(\lambda-\alpha)+1\right]\end{matrix}}r^2\leq 0<|z|=r<1\end{matrix}\qquad (3.1)$$

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The equality in (3.1) is attained for the function f, given by

$$f(z)=z-\frac{2(1-\beta)-(2\alpha-2\lambda)(k+\beta)}{\left[4(1+k)-(k+\beta)\right]\left[(2\lambda\alpha+\lambda-\alpha)+1\right]}.r^{2} \qquad (3.2)$$

Proof: Since $f\in TU_{s}(\lambda,\alpha,k,\beta)$ from in equation (2.1) we have

$$\sum_{n=2}^{\infty}\begin{bmatrix}\left[2n(1+k)-(k+\beta)\right]\left[(n-1)(2\lambda\alpha+\lambda-\alpha)+1\right]+\\(k+\beta)\left[n(n-1)\lambda\alpha-(\lambda-\alpha)((n+1)+1)(-1)^{n}\right]\end{bmatrix}a_{n}\le 2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]$$

It is easily known that

$$\begin{bmatrix}\left[4(1+1)-(k+\beta)\right]\left[(2\lambda\alpha+\lambda-\alpha)+1\right]\\(k+\beta)\left[2\lambda\alpha-3(\lambda-\alpha)+1\right]\end{bmatrix}\sum_{n=2}^{\infty}a_{n}\le$$

$$\sum_{n=2}^{\infty}\begin{bmatrix}\left[2n(1+k)-(k+\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda-\alpha)+1\right]+\\(k+\beta)\left[n(n-1)\lambda\alpha-(\lambda-\alpha)(n+1)+1\right](-1)^{n}a_{n}\end{bmatrix}$$

$$\le 2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]\ \forall\ 0<|z|=\lambda<1$$

$$\Rightarrow\sum_{n=2}^{\infty}a_{n}\le\frac{2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]}{\begin{bmatrix}4(1+k)-(k+\beta)\left[2\lambda\alpha+\lambda+\alpha+1\right]+\\(k+\beta)(2\lambda\alpha-3)(\lambda-\alpha)+1\end{bmatrix}} \qquad (3.3)$$

Consider

$$|f(z)|=\left|z-\sum_{n=2}^{\infty}a_{n}z^{n}\right|\le|z|+\sum_{n=2}^{\infty}a_{n}|z|^{n}.$$

$$\le r+\frac{2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]}{\begin{bmatrix}4(1+k)-(k+\beta)\left[2\lambda\alpha+\lambda+\alpha+1\right]+\\(k+\beta)(2\lambda\alpha-3(\lambda-\alpha)+1)\end{bmatrix}}.r^{2}$$

Using the in equality (3.3)

This gives the right hand side of in equality (3.1)

Also

$$|f(z)|=\left|z-\sum_{n=2}^{\infty}a_{n}z^{n}\right|\ge|z|+\sum_{n=2}^{\infty}a_{n}|z|^{n}$$

$$\ge r-r^{2}\sum_{n=2}^{\infty}a_{n}\ge r-\frac{2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]}{\begin{bmatrix}4(1+k)-(k+\beta)\left[2\lambda\alpha+\lambda+\alpha+1\right]+\\(k+\beta)(2\lambda\alpha-3(\lambda-\alpha)+1)\end{bmatrix}}.r^{2}$$

This is the left hand side of (3.1)

It can be easily seen that the function $f(z)$ defined by 3.2 is the extremal function for the theorem.

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Theorem 3.2: If the function $f \in TU_s(\lambda,\alpha,k,\beta)$, then

$$1-\frac{4\left[(1-\beta)-(2\alpha-2\lambda)(k+\beta)\right]}{\left[\left[\left(4(1+k)-(k+\beta)\right)\right]\left[(2\lambda\alpha+\lambda-\alpha)+1\right]\right]+(k+\beta)\left[2\lambda\alpha-3(\lambda-\alpha)+1\right]}r \le \left|f'(z)\right| \le$$
$$1+\frac{4\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]}{\left[\left[\left(4(1+k)-(k+\beta)\right)\right]\left[(2\lambda\alpha+\lambda-\alpha)+1\right]\right]+(k+\beta)\left[2\lambda\alpha-3(\lambda-\alpha)+1\right]}r \;\forall 0<|z|=r<1 \qquad (3.4)$$

The equality in (3.4) holds true for the function and given by (3.2)

Proof: Since $f \in TU_s(\lambda,\alpha,k,\beta)$ we have

$$\left|f'(z)\right| \le 1+\sum_{n=2}^{\infty} na_n |z|^{n-1} \le 1+r\sum_{n=2}^{\infty} na_n \qquad (3.5)$$

And

$$\left|f'(z)\right| \ge 1-\sum_{n=2}^{\infty} na_n |z|^{n-1} \ge 1-r\sum_{n=2}^{\infty} na_n \qquad (3.6)$$

The result in (3.4) holds true from (3.5), (3.6) and using the simple consequence of (3.3) given by

$$\sum_{n=2}^{\infty} na_n \le \frac{4\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]}{\left[\left[4(1+k)-(k+\beta)\right]\left[2\lambda\alpha+\lambda+\alpha+1\right]+(k+\beta)\left(2\lambda\alpha-3(\lambda-\alpha)+1\right)(-1)^n\right]}(-1)^n$$

The result is sharp for the function of given in 3.2.

4. Closure theorems for the class $TU_s(\lambda,\alpha,k,\beta)$

In the next theorems we prove that the class $f \in TU_s(\lambda,\alpha,k,\beta)$ is closed under convex linear combinations

Theorem 4.1: If $f_1(z)=z$ and

$$f_n(z)=z-\frac{2\left[(1-\beta)-(\lambda-\alpha)(k+\beta)\right]}{\left[\left[2n(1+k)-k+\beta\right]\right]\left[(n-1)(n\lambda\alpha+\lambda+\alpha)+1\right]+\left[(k+\beta)\left(n\lambda\alpha-(n+1)(\lambda-\alpha)+1\right)\right](-1)^n} \qquad (4.1)$$

Then $f \in TU_s(\lambda,\alpha,k,\beta)$ iff it can be expressed in the form $f(z)=\sum_{n=1}^{\infty}\lambda_n f_n(z)$ where $\lambda_n \ge 0$ and $\sum_{n=1}^{\infty}\lambda_n = 1$

Proof: Suppose $f(z)=\sum_{n=1}^{\infty}\lambda_n f_n(z)$ with $\lambda_n \ge 0$ and $\sum_{n=1}^{\infty}\lambda_n = 1$

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Since

$$\sum_{n=1}^{\infty}\lambda_n f_n(z)$$

$$=\lambda_1 f_1(z)+\sum_{n=2}^{\infty}\lambda_n f_n(z)$$

$$=\left[1-\sum_{n=2}^{\infty}\lambda_n\right]z+\sum_{n=2}^{\infty}\lambda_n\left[z-\frac{2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]}{\left[\begin{array}{l}\left[2n(1+k)-(k+\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda+\alpha)+1\right]\\+\left[(k+\beta)\left[n\lambda\alpha-(n+1)(\lambda-\alpha)+1\right](-1)^n\right]\end{array}\right]}z^n\right]$$

$$=z-\sum_{n=2}^{\infty}\lambda_n\left[\frac{2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]}{\left[\begin{array}{l}\left[2n(1+k)-(k+\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda+\alpha)+1\right]\\+\left[(k+\beta)\left[n\lambda\alpha-(n+1)(\lambda-\alpha)+1\right](-1)^n\right]\end{array}\right]}\right]z^n$$

Consider

$$\sum_{n=2}^{\infty}\lambda_n\frac{2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]}{\left[\begin{array}{l}\left[2n(1+k)-(k+\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda+\alpha)+1\right]\\+\left[(k+\beta)\left[n\lambda\alpha-(n+1)(\lambda-\alpha)+1\right](-1)^n\right]\end{array}\right]}\,x\,\frac{\left[\begin{array}{l}\left[2n(1+k)-(k+\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda+\alpha)+1\right]\\+\left[(k+\beta)\left[n\lambda\alpha-(n+1)(\lambda-\alpha)+1\right](-1)^n\right]\end{array}\right]}{2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]}$$

$$=\sum_{n=2}^{\infty}\lambda_n=1-\lambda_1\leq 1$$

Thus the coefficient of $f(z)$ satisfies inequality (2.1). Hence from theorem (2.2) it follows that $f\in TU_s(\lambda,\alpha,k,\beta)$.

Conversely suppose $f\in TU_s(\lambda,\alpha,k,\beta)$.

Since $a_n\leq\frac{2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]}{\left[\begin{array}{l}\left[2n(1+k)-(k+\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda+\alpha)+1\right]\\+\left[(k+\beta)\left[n\lambda\alpha-(n+1)(\lambda-\alpha)+1\right](-1)^n\right]\end{array}\right]}$, $n\geq 2$

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Setting $\lambda_n \le \frac{\begin{bmatrix}[2n(1+k)-(k+\beta)][(n-1)(n\lambda\alpha+\lambda+\alpha)+1] \\ +[(k+\beta)[n\lambda\alpha-(n+1)(\lambda-\alpha)+1](-1)^n]\end{bmatrix}}{2[(1-\beta)-(\alpha-\lambda)(k+\beta)]}$, a_n

And $\lambda_1 = 1-\sum_{n=2}^{\infty}\lambda_n$ then

$f(z)=\sum_{n=1}^{\infty}\lambda_n f_n(z)$

This completes the proof of the theorem.

Theorem 4.2: The class $TU_s(\lambda,\alpha,k,\beta)$ is closed under convex linear combinations.

Proof: Suppose that each of the function $f_l(z)$ given by $f_l(z)=z-\sum_{n=1}^{\infty}a_{n,l}z^n\ (l=1,2)$ is in the class $TU_s(\lambda,\alpha,k,\beta)$.

We need to prove that the function $H(z)$ given by

$H(z)=\lambda_2 f_1(z)+(1-\lambda)f_2(z)\ (0\le\lambda\le1)$ also lies in the class $TU_s(\lambda,\alpha,k,\beta)$.

Since $H(z)=z-\sum_{n=2}^{\infty}[\lambda a_{n,l}+(1-\lambda)a_{n,2}]z^n$

Consider

$$\sum_{n=2}^{\infty}\begin{bmatrix}[2n(1+k)-(k+\beta)][(n-1)(n\lambda\alpha+\lambda+\alpha)+1]+ \\ (k+\beta)[(n\lambda\alpha-(n+1)(\lambda-\alpha)+1)](-1)^n\end{bmatrix}[\lambda a_{n,l}+(1-\lambda)a_{n,2}]$$

$$=\lambda\sum_{n=2}^{\infty}\begin{bmatrix}[2n(1+k)-(k+\beta)][(n-1)(n\lambda\alpha+\lambda+\alpha)+1]+ \\ (k+\beta)[(n\lambda\alpha-(n+1)(\lambda-\alpha)+1)](-1)^n\end{bmatrix}a_{n,l}$$

$$+(1-\lambda)\sum_{n=2}^{\infty}\begin{bmatrix}[2n(1+k)-(k+\beta)][(n-1)(n\lambda\alpha+\lambda+\alpha)+1]+ \\ (k+\beta)[(n\lambda\alpha-(n+1)(\lambda-\alpha)+1)](-1)^n\end{bmatrix}a_{n,2}$$

$$\le\lambda\left(2[(1-\beta)-(\alpha-\lambda)(k+\beta)]+(1-\lambda)[2[(1-\beta)-(\alpha-\lambda)(k+\beta)]]\right)$$

$$\le\ 2[(1-\beta)-(\alpha-\lambda)(k+\beta)]$$

Thus from theorem 2.1, $H(z)\in TU_s(\alpha,\lambda,k,\beta)$

Hence the class $TU_s(\lambda,\alpha,k,\beta)$ is closed under convex linear combinations.

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5. Radii of Star likeness, Convexity and close to convexity for the functions in the class $TU_s(\lambda,\alpha,k,\beta)$

Theorem 5.1: If, $f(z)\in TU_s(\lambda,\alpha,k,\beta)$ then f is starlike of order $\delta(0\le\delta<1)$

$|z|<r_1(\lambda,\alpha,k,\beta,\delta)$ Where

$$r_1(\lambda,\alpha,k,\beta,\delta)=\inf_n \frac{\begin{bmatrix}(1-\delta)\left[2n(1+k)-(k+\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda+\alpha)+1\right]\\+\left[(k+\beta)\left[n\lambda\alpha-(n+1)(\lambda-\alpha)+1\right](-1)^n\right]\end{bmatrix}}{(n-\delta)2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]}\ \forall n\ge 2$$

And the result is sharp.

Proof: Suppose $f(z)\in TU_s(\lambda,\alpha,k,\beta)$. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}-1\right|\le 1-\delta\left[0\le\delta<1,|z|<r_1(\lambda,\alpha,k,\beta,\delta)\right] \quad (5.1)$$

Replacing $f(z)$, $zf'(z)$ in the left hand side of (5.1) with their equivalent expressions in series, we get

$$\left|\frac{\sum_{n=2}^{\infty}(n-1)a_n z^{n-1}}{1-\sum_{n=2}^{\infty}(n-1)a_n z^{n-1}}\right|\le\frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}$$

This will be bounded by $(1-\delta)$ if

$$\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}\le(1-\delta)\left[1-\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}\right]$$

$$\sum_{n=2}^{\infty}\left(\frac{n-\delta}{1-\delta}\right)|z|^{n-1}a_n\le 1 \quad (5.2)$$

Since for $f(z)\in TU_s(\lambda,\alpha,k,\beta)$ from theorem (2.1) we have

$$\sum_{n=2}^{\infty}\frac{\begin{bmatrix}\left[2n(1+k)-(k+\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda+\alpha)+1\right]\\+\left[(k+\beta)\left[n\lambda\alpha-(n+1)(\lambda-\alpha)+1\right](-1)^n\right]\end{bmatrix}}{2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]}a_n\le 1$$

The condition (5.2) will be satisfied if

$$\left(\frac{n-\delta}{1-\delta}\right)|z|^{n-1}\le\frac{\begin{bmatrix}\left[2n(1+k)-(k+\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda+\alpha)+1\right]\\+\left[(k+\beta)\left[n\lambda\alpha-(n+1)(\lambda-\alpha)+1\right](-1)^n\right]\end{bmatrix}}{2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]}\ \textit{for each } n\ge 2$$

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$$\Rightarrow |z| \leq \frac{\begin{bmatrix}(1-\delta)\left[2n(1+k)-(k+\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda+\alpha)+1\right] \\ +\left[(k+\beta)\left[n\lambda\alpha-(n+1)(\lambda-\alpha)+1\right](-1)^n\right]\end{bmatrix}}{(n-\delta)2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]}\frac{1}{n-1}$$

Setting $|z| = r_1(\lambda,\alpha,k,\beta,\delta)$, the result of the theorem follows. And the result is sharp for each n for the functions $f_n(z)$ given in (2.8).

Theorem 5.2: If, $f(z) \in TU_s(\lambda,\alpha,k,\beta)$ then f is convex of order $\delta(0 \leq \delta < 1)$ in $|z| < r_1(\lambda,\alpha,k,\beta,\delta)$ Where

$$r_2(\lambda,\alpha,k,\beta,\delta) = \inf_n \frac{\begin{bmatrix}(1-\delta)\left[2n(1+k)-(k+\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda+\alpha)+1\right] \\ +\left[(k+\beta)\left[n\lambda\alpha-(n+1)(\lambda-\alpha)+1\right](-1)^n\right]\end{bmatrix}}{2n(n-\delta)\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]} \quad \forall n \geq 2$$

And the result is sharp.

Proof: Suppose $f \in TU_s(\lambda,\alpha,k,\beta)$. It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq 1-\delta\left[for\ 0 \leq \delta < 1, |z| < r_2(\lambda,\alpha,k,\beta,\delta)\right] \quad (5.3)$$

Replacing $f'(z)$ and $f''(z)$ values in the left hand side of (5.3) with their equivalent expressions in series, then we get

$$\left|\frac{\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1-\sum_{n=2}^{\infty} na_n z^{n-1}}\right| \leq \frac{\sum_{n=2}^{\infty} n(n-1)a_n |z|^{n-1}}{1-\sum_{n=2}^{\infty} na_n |z|^{n-1}}$$

This will be bounded by $(1-\delta)$ if

$$\sum_{n=2}^{\infty} n(n-1)a_n |z|^{n-1} \leq (1-\delta)\left[1-\sum_{n=2}^{\infty} na_n |z|^{n-1}\right] \text{ or}$$

$$\sum_{n=2}^{\infty} \frac{n(n-\delta)}{(1-\delta)} a_n |z|^{n-1} \leq 1 \qquad (5.4)$$

Since for $f \in TU_s(\lambda,\alpha,k,\beta)$ from theorem (2.1) we have

$$\sum_{n=2}^{\infty} \frac{\begin{bmatrix}\left[2n(1+k)-(k+\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda+\alpha)+1\right] \\ +\left[(k+\beta)\left[n\lambda\alpha-(n+1)(\lambda-\alpha)+1\right](-1)^n\right]\end{bmatrix}}{2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]} a_n \leq 1$$

The condition (5.4) will be satisfied if

$$\frac{n(n-\delta)}{(1-\delta)}|z|^{n-1} \le \frac{\left[\begin{array}{l}\left[2n(1+k)-(k+\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda+\alpha)+1\right] \\ +\left[(k+\beta)\left[n\lambda\alpha-(n+1)(\lambda-\alpha)+1\right](-1)^n\right]\end{array}\right]}{2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]} \quad \text{for each } n \ge 2$$

$$\Rightarrow |z| \le \left[\frac{\begin{array}{l}(1-\delta)\left[2n(1+k)-(k+\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda+\alpha)+1\right] \\ +\left[(k+\beta)\left[n\lambda\alpha-(n+1)(\lambda-\alpha)+1\right](-1)^n\right]\end{array}}{2n(n-\delta)\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]}\right]^{\frac{1}{n-1}} \quad , n = 2,3$$

Setting $|z| = r_2(\lambda,\alpha,k,\beta,\delta)$, the result of the theorem follows.

Sharpness of this result can be easily verified for functions $f_n(z)$ stated as in (2.8).

Theorem 5.3: If, $f \in TU_s(\lambda,\alpha,k,\beta)$ then f is close to convex of order $\delta(0 \le \delta < 1)$ in $|z| < r_1(\lambda,\alpha,k,\beta,\delta)$ where

$$r_3(\lambda,\alpha,k,\beta,\delta) = \inf_n \left[\frac{\begin{array}{l}(1-\delta)\left[2n(1+k)-(k+\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda+\alpha)+1\right] \\ +\left[(k+\beta)\left[n\lambda\alpha-(n+1)(\lambda-\alpha)+1\right](-1)^n\right]\end{array}}{2n(n-\delta)\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]}\right]^{1/n-1} \quad \forall n \ge 2$$

And the result is sharp.

Proof: Suppose $f \in TU_s(\lambda,\alpha,k,\beta)$. It is sufficient to show that

$$|f'(z)-1| \le 1-\delta\left[\text{for } 0 \le \delta < 1, |z| < r_3(\lambda,\alpha,k,\beta,\delta)\right] \qquad (5.5)$$

Replacing $f'(z)$ in the left hand side of (5.5) with their equivalent expressions in series, then we get

$$\left|1-\sum_{n=2}^{\infty} na_n z^{n-1} - 1\right| \le \sum_{n=2}^{\infty} na_n |z|^{n-1}$$

This will be bounded by $(1-\delta)$ if

$$\sum_{n=2}^{\infty} na_n |z|^{n-1} \le 1-\delta \quad \text{or}$$

$$\sum_{n=2}^{\infty} \frac{n}{(1-\delta)} a_n |z|^{n-1} \le 1 \quad (5.6)$$

Since for $f \in TU_s(\lambda,\alpha,k,\beta)$ from theorem (2.2) we have

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$$\sum_{n=2}^{\infty} \frac{\begin{bmatrix} \left[2n(k+1)-(k+\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda+\alpha)+1\right] \\ +\left[(k+\beta)\left[n\lambda\alpha-(n+1)(\lambda-\alpha)+1\right](-1)^n\right] \end{bmatrix}}{2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]} a_n \le 1$$

The condition (5.6) will be satisfied by

$$\left(\frac{n}{1-\delta}\right)|z|^{n-1} \le \frac{\begin{bmatrix} \left[2n(1+k)-(k+\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda+\alpha)+1\right] \\ +\left[(k+\beta)\left[n\lambda\alpha-(n+1)(\lambda-\alpha)+1\right](-1)^n\right] \end{bmatrix}}{2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]} \quad for\ each\ n \ge 2$$

Or

$$|z| \le \left[\frac{\begin{bmatrix} (1-\delta)\left[2n(1+k)-(k+\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda+\alpha)+1\right] \\ +\left[(k+\beta)\left[n\lambda\alpha-(n+1)(\lambda-\alpha)+1\right](-1)^n\right] \end{bmatrix}}{2n(n-\delta)\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]}\right]^{\frac{1}{n-1}} \quad ,\forall n \ge 2$$

Setting $|z| = r_3(\lambda,\alpha,k,\beta,\delta)$, the result follows.

Sharpness of this result can be easily verified for functions $f_n(z)$ stated as in (2.8).

Theorem 5.4: If, $f(z) \in TU_s(\lambda,\alpha,k,\beta)$ then $\sum_{n=2}^{\infty}\left(\frac{n-\delta}{1-\delta}\right) a_n \le 1$

Where $\delta \le 1 - \frac{2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]}{\begin{bmatrix} \left[4(1+k)-(k+\beta)\right]\left[(2\lambda\alpha+\lambda+\alpha)+1\right] \\ +\left[(k+\beta)\left[2\lambda\alpha-3(\lambda-\alpha)+1\right]-2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]\right] \end{bmatrix}}$

And the result is sharp.

Proof: Suppose $f \in TU_s(\lambda,\alpha,k,\beta)$. It is sufficient to show that

$$\sum_{n=2}^{\infty}\left(\frac{n-\delta}{1\delta}\right) a_n \le 1 \qquad (5.7)$$

Since $f \in TU_s(\lambda,\alpha,k,\beta)$ from theorem (2.1) we have

$$\sum_{n=2}^{\infty} \frac{\begin{bmatrix} \left[2n(k+1)-(k+\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda+\alpha)+1\right] \\ +\left[(k+\beta)\left[n\lambda\alpha-(n+1)(\lambda-\alpha)+1\right](-1)^n\right] \end{bmatrix}}{2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]} a_n \le 1$$

The condition (5.7) will be satisfied by

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$$\left(\frac{n-\delta}{1-\delta}\right)\le\frac{\begin{bmatrix}\left[2n(1+k)-(k+\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda+\alpha)+1\right]\\+\left[(k+\beta)\left[n\lambda\alpha-(n+1)(\lambda-\alpha)+1\right](-1)^n\right]\end{bmatrix}}{2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]}\text{ for each } n$$

$$\delta\le\frac{\begin{bmatrix}\left[2n(1+k)-(k+\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda+\alpha)+1\right]\\+\left[(k+\beta)\left[n\lambda\alpha-(n+1)(\lambda-\alpha)+1\right](-1)^n\right]-2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]\end{bmatrix}}{\begin{bmatrix}\left[2n(1+k)-(k+\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda+\alpha)+1\right]\\+\left[(k+\beta)\left[n\lambda\alpha-(n+1)(\lambda-\alpha)+1\right](-1)^n\right]-2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]\end{bmatrix}}\forall n\ge 2$$

$$\delta\le 1-\frac{(n-1)2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]}{\begin{bmatrix}\left[2n(1+k)-(k+\beta)\right]\left[(n-1)(n\lambda\alpha+\lambda+\alpha)+1\right]\\+\left[(k+\beta)\left[n\lambda\alpha-(n+1)(\lambda-\alpha)+1\right](-1)^n\right]-2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]\end{bmatrix}}$$

$$=\psi(n) \qquad (5.8)$$

Since for $\psi(n)\ge\psi(2)\forall n\ge 2,\ x\ge 0,\ 0\le\beta<1,\ 0\le\alpha\le\lambda\le 1$

Putting $n=2$ in (5.8) we get

$$\delta\le 1-\frac{2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]}{\begin{bmatrix}\left[4(1+k)-(k+\beta)\right]\left[(2\lambda\alpha+\lambda+\alpha)+1\right]+\\\left[(k+\beta)(2\lambda\alpha-3(\lambda-\alpha)+1)\right]-2\left[(1-\beta)-(\alpha-\lambda)(k+\beta)\right]\end{bmatrix}}$$

Hence the result. The sharpness of this result can be easily verified with the extremal function given in (3.2). This completes the proof of the theorem.

6. Integral operators

In the next theorem we consider the integral operators of functions in the class $TU_s(\lambda,\alpha,k,\beta)$.

Theorem 6.1: If, $f\in TU_s(\lambda,\alpha,k,\beta)$ then the function defined by

$$F(z)=\frac{1+C}{z^c}\int_0^c t^{c-1}f(z)\,dt\ \left[(t>-1)\right] \text{ is also} \qquad (6.1)$$

in $TU_s(\lambda,\alpha,k,\beta)$

Proof: Suppose $f(z)\in TU_s(\lambda,\alpha,k,\beta)$ from (6.1) we have

$$f(z)=z-\sum_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)a_n z^n,\ 0<\frac{c+1}{c+n}<1$$

Consider

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$$\sum_{n=2}^{\infty}\begin{bmatrix}[2n(1+k)-(k+\beta)][(n-1)(n\lambda\alpha+\lambda+\alpha)+1]+\\(k+\beta)(n\lambda\alpha-(n+1)(\lambda-\alpha)+1)(-1)^n\end{bmatrix}a_n\left(\frac{C+1}{C+n}\right)$$

$$\leq\begin{bmatrix}[2n(1+k)-(k+\beta)][(n-1)(n\lambda\alpha+\lambda+\alpha)+1]+\\(k+\beta)(n\lambda\alpha-(n+1)(\lambda-\alpha)+1)(-1)^n\end{bmatrix}a_n$$

$$\leq\begin{bmatrix}[2n(1+k)-(k+\beta)][(n-1)(n\lambda\alpha+\lambda+\alpha)+1]+\\(k+\beta)(n\lambda\alpha-(n+1)(\lambda-\alpha)+1)(-1)^n\end{bmatrix}a_n$$

$\leq 2[(1-\beta)-(\alpha-\lambda)(k+\beta)]$ (Using inequality (2.1)

$\Rightarrow f(z)\in TU_s(\lambda,\alpha,k,\beta)$ Hence the theorem.

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