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COEFFICIENT INEQUALITIES FOR CERTAIN SUBCLASSES OF p-VALENT FUNCTIONS

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Abstract:

The object of this paper is to introduce a subclass of p-valent functions and obtain the coefficient inequalities. Fekete-Szego inequality for the functions in this class.2000 Mathematic subject classification. Primary 30C45.

Key Words: p-valent functions, convex, starlike functions.

1. Introduction

Let $\mathcal{A}_p$ denote the class of all p-valent functions f of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad \ldots \qquad (1.1)$$

in the open unit disc $U = \{z \in C : |z| < 1\}$

Here $\mathcal{A}_1 = \mathcal{A}$ and $p \in N$.

Let $S_p^*(\beta)$ and $C_p(\beta)$ be the classes consisting of the functions $f \in \mathcal{A}_p$ and satisfying the condition

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \beta \qquad \text{and}$$

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta \qquad \text{respectively}$$

for some $\beta (0 \le \beta < p)$, $z \in U$.

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These classes are known as p-valent starlike functions of order and p-valent convex functions of order respectively.

Several authors ([1][2][3][4]) have studied various subclasses of p-valent functions and obtained coefficient inequalities for the functions $f \in \mathcal{A}_p$ in these classes.

In this present paper we define a generalized subclass of p-valent function and study the coefficient inequalities and Fekete-Szego inequality for the functions in this class. The results of this paper generalize and unify several earlier results in this direction.

Definition 1.1: Let $Rs_p\,(\alpha, \lambda, \beta)$ be the class of all functions $f \in \mathcal{A}_p$ satisfying the condition.

$$Re\left(\frac{zf'(z)+\frac{\alpha}{\alpha+\lambda}z^2 f''(z)}{\left(1-\frac{\alpha}{\alpha+\lambda}\right) f(z)+\left(\frac{\alpha}{\alpha+\lambda}\right) zf'(z)}\right) > \beta, \qquad \forall z \in U \qquad (1.2)$$

Here $\alpha \geq 0,\ \lambda \geq 0,\ (\alpha, \lambda) \neq (0, 0)\ \ 0 \leq \beta < p$ and p is a fixed positive integer.

It is noted that

$Rs_p\,(0, 1, \beta) = S_p^*(\beta)$ and $Rs_p\,(1, 0, \beta) = C_p(\beta)$ are the usual classes of p-valent starlike and convex functions of orderβ.

To prove our results we require the following lemma.

Lemma (1.1) [1]: If $p(z) = 1 + c_1\, z + c_2\, z^2 + \ldots\ldots$ is a function with positive real part then for any complex number ν, we have

$$\left| c_2 - \nu c_1^2 \right| \leq 2 \text{ Max } \{1, |\,2\nu - 1|\,\}$$

This result is sharp for the functions

$$p(z) = \frac{1+z^2}{1-z^2} \text{ and } p(z) = \frac{1+z}{1-z}$$

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In the next sections we obtain the coefficient inequality and Fekete-Szego inequality for the function f in the class $Rs_p\ (\alpha,\lambda,\beta)$.

2. Coefficient Inequalities

In this section we first obtain a sufficient condition for the functions $f(z)\in \mathcal{A}p$ to be in the class $Rs_p\ (\alpha,\lambda,\beta)$.

Theorem 2.1: If $f(z)\in \mathcal{A}p$ and satisfies

$$\sum_{n=1}^{\infty}[\alpha(n+p)+\lambda]\,[n+p-\beta]\,|a_{n+p}|\le[p-\beta][\alpha p+\lambda] \qquad \ldots \qquad (2.1)$$

For $\alpha\ge 0,\ \lambda\ge 0,\ (\alpha,\ \lambda)\ne(0,0)$ and $0\le\beta<p$ then $f(z)\in RS_p\ (\alpha,\ \lambda,\ \beta)$.

Proof: Let $f(z)$ satisfies the condition in (2.1).

To prove $f(z)\in RS_p\ (\alpha,\ \lambda,\ \beta)$.

Using the technique

$\operatorname{Re}(w)>\beta \Leftrightarrow \dfrac{|w-p|}{|w+p-2\beta|}<1$, from the definition (1.1) consider

$$\frac{\left|\dfrac{zf'(z)+\left(\dfrac{\alpha}{\alpha+\lambda}\right)z^2 f''(z)}{\left(1-\dfrac{\alpha}{\alpha+\lambda}\right)f(z)+\left(\dfrac{\alpha}{\alpha+\lambda}\right)zf'(z)}-p\right|}{\left|\dfrac{zf'(z)+\left(\dfrac{\alpha}{\alpha+\lambda}\right)z^2 f''(z)}{\left(1-\dfrac{\alpha}{\alpha+\lambda}\right)f(z)+\left(\dfrac{\alpha}{\alpha+\lambda}\right)zf'(z)}+p-2\beta\right|} \qquad \ldots\ldots (2.2)$$

Replacing the values of $f(z)$, $f'(z)$ and $f''(z)$ with their equivalent series expressions in (2.2) and after simplification. We have

$$\frac{\left|(\alpha+\lambda)\left[pz^p+\sum_{n=1}^{\infty}a_{n+p}(n+p)z^{n+p}\right]+\alpha\left[p(p-1)z^p+\sum_{n=1}^{\infty}a_{n+p}(n+p)(n+p-1)z^{n+p}\right]-p\lambda\left[z^p+\sum_{n=1}^{\infty}a_{n+p}z^{n+p}\right]-\alpha p\left[pz^p+\sum_{n=1}^{\infty}a_{n+p}(n+p)z^{n+p}\right]\right|}{\left|(\alpha+\lambda)\left[pz^p+\sum_{n=1}^{\infty}a_{n+p}(n+p)z^{n+p}\right]+\alpha\left[p(p-1)z^p+\sum_{n=1}^{\infty}a_{n+p}(n+p)(n+p-1)z^{n+p}\right]+\lambda(p-2\beta)\left[z^p+\sum_{n=1}^{\infty}a_{n+p}z^{n+p}\right]+\alpha(p-2\beta)\left[pz^p+\sum_{n=1}^{\infty}a_{n+p}(n+p)z^{n+p}\right]\right|}$$

$$=\frac{\left|\left[p(\alpha+\lambda)+p(p-1)\alpha-p\lambda-p^2\alpha\right]z^p+\sum_{n=1}^{\infty}\left[(\alpha+\lambda)(n+p)+\alpha(n+p)(n+p-1)-p\lambda-\alpha p(n+p)\right]a_{n+p}z^{n+p}\right|}{\left|\left[p(\alpha+\lambda)+\alpha p(p-1)+\lambda(p-2\beta)+\alpha p(p-2\beta)\right]z^p+\sum_{n=1}^{\infty}\left[(\alpha+\lambda)(n+p)+\alpha(n+p)(n+p-1)+\lambda(p-2\beta)+\alpha(p-2\beta)(n+p)\right]a_{n+p}z^{n+p}\right|}.$$

$$=\frac{\left|\sum_{n=1}^{\infty}\left[n\alpha(n+p)+n\lambda\right]a_{n+p}z^{n+p}\right|}{\left|(p-\beta)(2\alpha p+2\lambda)z^p+\sum_{n=1}^{\infty}\left[n+2p-2\beta\right]\left[\alpha(n+p)+\lambda\right]a_{n+p}z^{n+p}\right|}$$

$$\leq\frac{\sum_{n=1}^{\infty}\left[n\alpha(n+p)+n\lambda\right]\left|a_{n+p}\right|\left|z\right|^{n+p}}{(p-\beta)(2\alpha p+2\lambda)\left|z\right|^p-\sum_{n=1}^{\infty}\left[n+2p-2\beta\right]\left[\alpha(n+p)+\lambda\right]\left|a_{n+p}\right|\left|z\right|^{n+p}}$$

This will be bounded by 1 if

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$$\sum_{n=1}^{\infty}\left[n\alpha\,(n+p)+n\lambda\right]\left|a_{n+p}\right| \le (p-\beta)\,(2\alpha p+2\lambda)-\sum_{n=1}^{\infty}\left[n+2p-2\beta\right]\left[\alpha(n+p)+\lambda\right]\left|a_{n+p}\right|$$

$$\Rightarrow \sum_{n=1}^{\infty}\left[\alpha\,(n+p)+\lambda\right]\left[n+p-\beta\right]\left|a_{n+p}\right|-(p-\beta)\,(\alpha p+\lambda)\le 0$$

Hence $f(z)\in RS_p\,(\alpha,\ \lambda,\ \beta)$. This completes the proof of the theorem (2.1)

Theorem 2.2: If $f\ (z)\in \mathcal{A}_p$ is in the class $Rs_p\,(\alpha,\lambda,\beta)$ then

$$|a_{p+1}|\le \frac{2\,(p-\beta)}{\left[\alpha\,(p+1)+\lambda\right]}\left[\alpha p+\lambda\right] \qquad (2.3)$$

$$\left|a_{p+n}\right|\le \frac{2\,(p-\beta)(\alpha p+\lambda)}{n\left[\alpha\,(p+n)+\lambda\right]}\prod_{j=1}^{n-1}\left[1+\frac{2(p-\beta)}{j}\right] \qquad \forall\, n\ge 2 \qquad (2.4)$$

Proof: Since $f\ (z)\in Rs_p\ (\alpha,\lambda,\beta)$ from the definition (1.1) we have

$$\mathrm{Re}\left(\frac{zf'(z)+\dfrac{\alpha}{\alpha+\lambda}z^2 f''(z)}{\left(1-\dfrac{\alpha}{\alpha+\lambda}\right)f(z)+\left(\dfrac{\alpha}{\alpha+\lambda}\right)zf'(z)}\right)>\beta$$

Define a function $p\,(z)$ such that

$$p(z)=\frac{\dfrac{zf'(z)+\left(\dfrac{\alpha}{\alpha+\lambda}\right)z^2 f''(z)}{\left(1-\dfrac{\alpha}{\alpha+\lambda}\right)f(z)+\left(\dfrac{\alpha}{\alpha+\lambda}\right)zf'(z)}-\beta}{p-\beta}=1+\sum_{n=1}^{\infty}c_n\,z^n \quad \ldots..\ (2.5)$$

Here $p\,(z)$ is analytic in U with $p\,(0)=1$ and $\mathrm{Re}\left[p\,(z)\,\right]>0$. Upon simplifying the equation (2.5) we get

$$(p-\beta)\left[1+\sum_{n=1}^{\infty}c_n\,z^n\right]\left[\lambda\,f(z)+\alpha\,zf'(z)\right]=(\alpha+\lambda-\beta\alpha)\,zf'(z)+\alpha z^2 f''(z)-\beta\lambda\,f(z)$$

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...... (2.6)

Replacing $f(z)$, $f'(z)$ and $f''(z)$ with their equivalent expressions in series on both sides, after simplification we get.

$$[p-\beta]\left[\lambda \sum_{n=1}^{\infty} a_{n+p}\, z^{n+p} + \lambda \sum_{n=1}^{\infty} c_n\, z^{n+p}\right] + \lambda\left[\sum_{n=1}^{\infty} c_n\, z^n\right]\left[\sum_{n=1}^{\infty} a_{n+p} z^{n+p}\right]$$

$$+\alpha\left[\sum_{n=1}^{\infty} a_{n+p}\,(n+p)\,z^{n+p}\right] + \alpha p\left[\sum_{n=1}^{\infty} c_n\, z^{n+p}\right] + \alpha\left[\sum_{n=1}^{\infty} c_n z^n\right]\left[\sum_{n=1}^{\infty} a_{n+p}(n+p)z^{n+p}\right]$$

$$=(\alpha+\lambda-\beta\alpha)\left[pz^p + \sum_{n=1}^{\infty} a_{n+p}\,(n+p)\,z^{n+p}\right] + \alpha\left[p(p-1)\,z^p + \sum_{n=1}^{\infty} a_{n+p}\,(n+p)(n+p-1)\,z^{n+p}\right]$$

$$-\beta\lambda\left[z^p + \sum_{n=1}^{\infty} a_{n+p}\quad z^{n+p}\right] \quad ..... (2.7)$$

Comparing the coefficient of z^{n+p} on both sides of (2.7) we get

$$a_{n+p} = \frac{[p-\beta]}{n[\alpha(n+p)+\lambda]}\{(\alpha p+\lambda)c_n + [\lambda+\alpha\,(p+1)]c_{n-1}\, a_{p+1}$$

$$+[\lambda+\alpha\,(p+2)]c_{n-2}\, a_{p+2} + [\lambda+\alpha\,(p+3)]c_{n-3}\, a_{p+3} + \ldots\ldots\ldots\ldots$$

$$\ldots\ldots + [\lambda+\alpha\,(n+p-2)]c_2\, a_{n+p-2} + [\lambda+\alpha\,(n+p-1)\,]c_1\, a_{n+p-1}\}$$

..... (2.8)

Taking modulus on both sides of (2.8) and applying the Carathedory inequality $|c_n| \le 2\ \forall n \ge 1$ we get

$$|\,a_{n+p}| \le \frac{2(p-\beta)}{n\,[\alpha\,(n+p)+\lambda]}\{\,(\alpha p+\lambda) + [\lambda+\alpha\,(p+1)]\,|\,a_{p+1}| +$$

$$[\lambda+\alpha\,(p+2)]|\,a_{p+2}| + \ldots\ldots + [\lambda+\alpha\,(n+p-2)]|\,a_{n+p-2}| +$$

$$[\lambda+\alpha\,(n+p-1)]|\,a_{n+p-1}|\,\} \quad ..... (2.9)$$

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For $n = 1$, $\quad |a_{p+1}| \leq \frac{2(p-\beta)}{[\alpha(p+1)+\lambda]}[\alpha p + \lambda]$

This proves the result (2.3), for $n = 2$,

$$|a_{p+2}| \leq \frac{2(p-\beta)}{2[\alpha(p+2)+\lambda]}[(\alpha p+\lambda)\,[1+2(p-\beta)]]$$

The result in (2.4) is true for $n = 2$

Suppose that the result in (2.4) is true for $3 \leq n \leq m$.

For $n = m+1$ consider

$$|a_{p+m+1}| \leq \frac{2(p-\beta)}{(m+1)[\alpha(p+m+1)+\lambda]}\Big\{(\alpha p+\lambda)+2(p-\beta)[\alpha p+\lambda]+ \frac{2(p-\beta)}{2}[(\alpha p+\lambda)(1+2(p-\beta))]+\ldots\ldots\ldots + \ldots\ldots\ldots\frac{2(p-\beta)}{m}(\alpha p+\lambda)\pi_{j=1}^{m-1}\left[1+\frac{2(p-\beta)}{j}\right]\Big\}$$

$$|a_{p+m+1}| \leq \frac{2(p-\beta)}{(m+1)[\alpha(p+m+1)+\lambda]}(\alpha p+\lambda)\pi_{j=1}^{m}\left[1+\frac{2(p-\beta)}{j}\right]$$

Thus the result in (2.4) is true for $n = m+1$. Hence by Mathematical induction the result is true for all values of $n \geq 1$. This completes the proof of the theorem.

3. Fekete – Szego Inequality

Theorem 3.1: If $f(z) \in \mathcal{A}_p$ is in the class $Rs_p[\alpha, \lambda, \beta]$ then for any complex number μ, we have

$$|a_{p+2} - \mu a_{p+1}^2| \leq \frac{[p-\beta][\alpha p+\lambda]}{[\alpha(p+2)+\lambda]}\max\left\{1, \frac{|4(p-\beta)(\alpha p+\lambda)[\alpha(p+2)+\lambda]\mu-\delta|}{[\alpha[p+1]+\beta]^2}\right\}$$

Where $\delta = [2(p-\beta)+1][\alpha(p+1)+\lambda]^2$ ..... (3.1)

And the result is sharp

Proof: Since $f(z) \in Rs_p[\alpha, \lambda, \beta]$ from equation (2.8) we have

$$a_{p+1} = \frac{(p-\beta)}{[\alpha(p+1)+\lambda]}[\alpha p+\lambda]c_1 \text{ and}$$

$$a_{p+2} = \frac{[p-\beta]}{2[\alpha(p+2)+\lambda]}\left[(\alpha p+\lambda)c_2 + [\alpha(p+1)+\lambda]c_1 a_{p+1}\right]$$

$$a_{p+2} = \frac{[p-\beta]}{2[\alpha(p+2)+\lambda]}[\alpha p+\lambda]\left[c_2 + (p-\beta)c_1^2\right].$$

For any complex number μ, we have

$$a_{p+2} - \mu a_{p+1}^2 = \frac{[p-\beta][\alpha p+\lambda]}{2[\alpha(p+2)+\lambda]}\left[c_2 + (p-\beta)c_1^2\right] - \frac{\mu[p-\beta]^2[\alpha p+\lambda]^2}{[\alpha(p+1)+\lambda]^2}c_1^2$$

$$a_{p+2} - \mu a_{p+1}^2 = \frac{[p-\beta][\alpha p+\lambda]}{2[\alpha(p+2)+\lambda]}\left[c_2 - \nu c_1^2\right] \quad (3.2)$$

Where $\nu = \frac{2[p-\beta][\alpha p+\lambda][\alpha(p+2)+\lambda]}{[\alpha(p+1)+\lambda]^2}\mu - [p-\beta]$

Taking modulus on both sides of (3.2) and applying Lemma (1.1) on R.H.S we get

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \le \frac{[p-\beta][\alpha p+\lambda]}{[\alpha(p+2)+\lambda]}\max\left\{1, |2\nu-1|\right\}$$

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \le \frac{[p-\beta][\alpha p+\lambda]}{[\alpha(p+2)+\lambda]}\max\left\{1, \left|\frac{4[\alpha(p+2)+\lambda](p-\beta)(\alpha p+\lambda)\mu - \delta}{[\alpha[p+1]+\lambda]^2}\right|\right\}$$

Where $\delta = [2(p-\beta)+1][\alpha(p+1)+\lambda]^2$

This completes the proof of the theorem. The result is sharp.

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$$\left|a_{p+2} - \mu a_{p+1}^{2}\right| = \frac{[p - \beta]\,[\alpha p + \lambda]}{[\alpha\,(p+2) + \lambda]}, \qquad \text{if } p(z) = \frac{1+z^{2}}{1-z^{2}}$$

$$\left|a_{p+2} - \mu a_{p+1}^{2}\right| = \frac{[p - \beta]\,[\alpha p + \lambda]}{[\alpha\,(p+2) + \lambda]} \left| \frac{4\,[\alpha(p+2) + \lambda](p-\beta)\,(\alpha p + \lambda)\,\mu - \delta}{[\alpha\,[p+1] + \lambda]^{2}} \right|,$$

if $\quad p(z) = \dfrac{1+z}{1-z}$

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