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Coefficient Inequalities for Transforms of Analytic Functions with Negative Coefficients

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Abstract

In this paper we introduce a subclass of analytic functions associated with k^{th} root transforms. We study the coefficient bounds, distortion properties, extreme points, radius of starlikeness, convexity, close to convexity and integral transformations for the function f in this class. The results of this paper generalize many earlier results in this direction.

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1. Introduction

Let A be the class of all functions f analytic in the open unit disc $\Delta = [z \in C : |z| < 1]$ normalized by $f(0)=0$ and $f'(0)=1$. Let f be a function in the class A of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n; \ \forall z \in \Delta \quad (1)$$

Let S be the subclass of A consisting of univalent functions. Let $S^*(\beta)$ and $C(\beta)$ be the classes of functions starlike of order β and convex of order $\beta(0 \le \beta \le 1)$ respectively, defined as follows

$$\Re e\left\{\frac{zf'(z)}{f(z)}\right\} > \beta$$

$$\Re e\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta$$

Let T be the subclass of S consisting of function f of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n; \quad a_n \ge 0 \quad (2)$$

A function $f \in T$ is called as a function with negative coefficients and introduced by Silverman [10]. He investigated the starlike and convex functions of order β with negative coefficients. These classes are denoted by $S_T^*(\beta)$ and $C_T(\beta)$ respectively. Goodman [2, 3] introduced the concept of uniform starlikeness and uniform convexity for functions in A .A function f is said to be uniformly convex if f is convex and has the property that each circular arc γ contained in Δ, with center ξ is also in Δ, the arc $f(\gamma)$ is convex. Similarly the function f is uniformly starlike if f is starlike and has the property that for each circular arc γ is contained in Δ with center ξ is also in Δ, the arc $f(\gamma)$ is starlike. The classes of functions consisting of uniformly convex and starlike functions are denoted by UCV and UST respectively.

The following analytic characterization of UCV and UST are obtained by Goodman[2,3]. The class of uniformly convex functions (UCV) consists of functions $f \in A$ satisfying

$$\Re e\left\{1+(z-\xi)\frac{f''(z)}{f'(z)}\right\} \geq 0, \quad \forall z,\xi \in \Delta \tag{3}$$

The class of uniformly starlike functions (UST) consists of functions $f \in A$ satisfying

$$\Re e\left\{\frac{(z-\xi)f'(z)}{f(z)-f(\xi)}\right\} \geq 0, \quad \forall z,\xi \in \Delta \tag{4}$$

Ronning [7], Ma and Minda [5] have individually given the following one variable characterization for the function f in UCV and UST classes.

A function $f \in A$ is said to be in the class UCV if and only if

$$\Re e\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right| \quad \forall z \in \Delta \tag{5}$$

Let the class of functions f for which there is a uniformly convex function F such that $f(z)=zF'(z)$, is denoted by S_p. It is easy to see that the function f is in S_p if and only if

$$\Re e\left\{\frac{zf'(z)}{f(z)}\right\} > \left|\frac{zf'(z)}{f(z)}-1\right| \quad \forall z \in \Delta \tag{6}$$

Recently many research workers has extended or generalized the classes (UST), UCV and the class S_p. Recently S.Shams, S.R.Kulkarni and J.M.Jahangiri [9] introduced the classes $SD(k,\beta)$ and $KD(k,\beta)$ to be the classes of functions. The k^{th} root transformation of an analytic function $f(z) \in A$ is given by $f \in A$ satisfying the conditions

$$\Re e\left\{\frac{zf'(z)}{f(z)}\right\} > k\left|\frac{zf'(z)}{f(z)}-1\right| + \beta$$

$$\Re e\left\{1+\frac{zf''(z)}{f'(z)}\right\} > k\left|\frac{zf''(z)}{f'(z)}\right| + \beta$$

respectively for some $k \geq 0$ and $\beta\ (0 \leq \beta < 1)$. It is noted that $f(z) \in KD(k,\beta)$ if and only if $f(z) \in SD(k,\beta)$.they have shown some sufficient conditions for f to be in the classes $SD(k,\beta)$ and $KD(k,\beta)$.

By imposing the condition $0 \leq k \leq \beta$, S.Owa, Y.Polatoglu and E.Yavuz [6] obtained coefficient inequalities, distortion properties for the functions in the classes $SD(k,\beta)$ and $KD(k,\beta)$.

Srivastava, Shanmugam, Ramchandran and Sivasubramanian [11] defined and studied the class $U(\lambda,\alpha,\beta,\kappa)$ to be the lass of functions $f \in T$ for which

$$\Re e\left\{\frac{zF'(z)}{F(z)}\right\} > k\left|\frac{zF'(z)}{F(z)}-1\right| + \beta$$

$(0 \geq \alpha \leq \lambda \leq 1), (0 \leq \beta < 1)$ and $k \geq 0$, where

$$F(z) = \lambda\alpha z^2 f''(z) + (\lambda-\alpha)zf'(z) + (1-\lambda+\alpha)f(z)$$

They have obtained the coefficient inequalities, necessary and sufficient conditions, distortion properties, convex linear combinations, radius of starlikeness, convexity and integral operators for the functions in this class.

$$T^k(f(z)) = \left[f\left(z^k\right)\right]^{\frac{1}{k}} = z + \sum_{n=1}^{\infty} b_{nk+1} z^{nk+1} \tag{7}$$

Here for $k=1$, $T^k(f(z)) = f(z)$.Also the k^{th} root transformation of an analytic function $f \in T$ is given by

$$T^k(f(z)) = z - \sum_{n=1}^{\infty} b_{nk+1} z^{nk+1}; \quad b_{nk+1} \geq 0 \ \forall n \tag{8}$$

In the present paper we define a subclass of analytic functions associated with the k^{th} root transformation and study the necessary and sufficient conditions, coefficient bounds, distortion properties, radius of starlikeness, convexity and integral transformations for the function in this class.

Definition 1.1 Let $R(\lambda,\alpha,\kappa,\beta)$ be the class of functions $T^k f \in T$ satisfying the condition

$$\Re e\left\{\frac{z\left[T^k(f(z))\right]' + \lambda z^2\left[T^k(f(z))\right]''}{T^k(f(z))}\right\} > \alpha\left|\frac{z\left[T^k(f(z))\right]' + \lambda z^2\left[T^k(f(z))\right]''}{T^k(f(z))} - 1\right| + \beta \tag{9}$$

For some $\lambda \geq 0,\ \alpha \geq 0,\ k \geq 1$ and $0 \leq \beta < 1$.

Remarks: Here $R_T(\lambda,\alpha,k,\beta) = R(\lambda,\alpha,k,\beta) \cap T$. It can be seen that

1. $R_T(\lambda,\alpha,1,\beta) = R_T(\lambda,\alpha,\beta)$, defined and studied by K.Saroja [8].
2. $R_T(0,\alpha,k,0)$ gives a class $USF(\alpha)$ associated with the k^{th} root transformationof f .
3. $R_T(0,\alpha,1,0) = USF(\alpha)$ defined and studied by S.Kanas and A.Wisniowska [12].
4. $R_T(0,0,k,\beta)$ gives a class $S_T^*(\beta)$ functions associated with the k^{th} root transformation of $f(z)$.
5. $R_T(0,0,1,\beta) = S_T^*(\beta)$ defined and studied by H.Silverman [10].

2. A Characterization theorem and resulting coefficient estimates

We first find a sufficient condition for the functions $T^k f(z) \in A$ to be in the class $R(\lambda,\alpha,k,\beta)$.

We give characterization of the class $R_T(\lambda,\alpha,k,\beta)$ by finding a necessary and sufficient condition for the function f to be in $R_T(\lambda,\alpha,k,\beta)$. This characterization also yields coefficient estimates for the function in this class.

Theorem 2.1 If $T^k f(z) \in A$ and satisfies the condition

$$\sum_{n=1}^{\infty}\left[(nk+1)(1+\lambda nk)(1+\alpha) - (\alpha+\beta)\right]\left|b_{nk+1}\right| \leq 1-\beta \tag{10}$$

for some $\lambda \geq 0, \alpha \geq 0, k \geq 1$and $0 \leq \beta < 1$then $T^k f(z) \in R(\lambda,\alpha,k,\beta)$.

Proof: Let $T^k f(z) \in A$ and satisfies the condition (10). To prove that $T^k f(z)$ is in the class $R(\lambda,\alpha,k,\beta)$. Applying the principle

$$\begin{aligned}&\Re e(w) > \alpha|w-1| + \beta \\ &\Leftrightarrow \Re e\left(w\left(1+\alpha e^{i\theta}\right) - \alpha e^{i\theta}\right) > \beta\ (-\pi \leq \theta \leq \pi, 0 \leq \beta < 1, \alpha \geq 0)\end{aligned} \tag{11}$$

For the function $w(z) = \dfrac{z\left[T^k(f(z))\right]' + \lambda z^2\left[T^k(f(z))\right]''}{T^k(f(z))}$ on R.H.S we get

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$$\Re e\left\{\frac{\{z[T^k(f(z))]'+\lambda z^2[T^k(f(z))]''\}\{1+\alpha e^{i\theta}\}-\alpha e^{i\theta}T^k(f(z))}{T^k(f(z))}\right\}>\beta \quad (12)$$

By setting $G(z)=\{z[T^k f(z)]'+\lambda z^2[T^k f(z)]''\}\{1+\alpha e^{i\theta}\}-\alpha e^{i\theta}T^k f(z)$ the above inequality (12) becomes

$|G(z)+(1-\beta)T^k(f(z))|>|G(z)-(1+\beta)T^k(f(z))|$ (13)

Replacing $T^k(f(z))$, $z[T^k(f(z))]'$ and $z^2[T^k(f(z))]''$ with their equivalent series expansions in (13), we get

$|G(z)+(1-\beta)T^k(f(z))|$

$$=\left|(2-\beta)z+\sum_{n=1}^{\infty}\{[(nk+1)(1+\lambda nk)+(1-\beta)]+\alpha e^{i\theta}[(nk+1)(1+\lambda nk)-1]\}b_{nk+1}z^{nk+1}\right|$$

$$\geq(2-\beta)|z|-\sum_{n=1}^{\infty}[(nk+1)(1+\lambda nk)+(1-\beta)]|b_{nk+1}||z^{nk+1}|-\alpha\sum_{n=1}^{\infty}[(nk+1)(1+\lambda nk)-1]\,|b_{nk+1}||z^{nk+1}| \quad (14)$$

Similarly we obtain

$$|G(z)-(1+\beta)T^k(f(z))|=\left|\beta z-\sum_{n=1}^{\infty}\{[(nk+1)(1+\lambda nk)-(1+\beta)]+\alpha e^{i\theta}[(nk+1)(1+\lambda nk)-1]\}b_{nk+1}z^{nk+1}\right|$$

$$\leq\beta|z|+\sum_{n=1}^{\infty}[(nk+1)(1+\lambda nk)-(1+\beta)]|b_{nk+1}||z^{nk+1}|+\alpha\sum_{n=1}^{\infty}[(nk+1)(1+\lambda nk)-1]|b_{nk+1}||z^{nk+1}| \quad (15)$$

Therefore from the inequalities (12) & (13), we have

$|G(z)+(1-\beta)T^k(f(z))|-|G(z)-(1+\beta)T^k(f(z))|$

$$\geq 2(1-\beta)-2\sum_{n=1}^{\infty}[(nk+1)(1+\lambda nk)(1+\alpha)-(\alpha+\beta)]|b_{nk+1}|$$

≥ 0 (Using the result in (10))

$$\Re e\left\{\frac{z[T^k(f(z))]'+\lambda z^2[T^k(f(z))]''}{[T^k(f(z))]}\right\}>k\left|\frac{z[T^k(f(z))]'+\lambda z^2[T^k(f(z))]''}{[T^k(f(z))]}-1\right|+\beta$$

Hence $[T^k(f(z))]\in R(\lambda,\alpha,k,\beta)$

Theorem 2.2 A necessary and sufficient condition for a function $T^k(f(z))\in T$ to be in the class $R_T(\lambda,\alpha,k,\beta)$ is that

$$\sum_{n=1}^{\infty}[(nk+1)(1+\lambda nk)(1+\alpha)-(\alpha+\beta)]b_{nk+1}\leq(1-\beta)$$

for some $\lambda\geq 0$, $\alpha\geq 0$, $k\geq 1$ and $0\leq\beta<1$.

Proof: In view of Theorem (2.1) it is sufficient to show that $T^k(f(z))$ satisfies the condition (8).

Suppose that $T^k(f(z))=z-\sum_{n=1}^{\infty}b_{nk+1}z^{nk+1}$ is in $R_T(\lambda,\alpha,k,\beta)$.

By setting $0\leq|z|=r<1$ and choosing the values of z on the real axis then from the inequality (10), we have

$$\Re e\left\{\frac{r-\sum_{n=1}^{\infty}\{nk(1+\lambda(nk+1))+\alpha e^{i\theta}[nk(1+\lambda(nk+1))]-1\}b_{nk+1}r^{nk+1}}{r-\sum_{n=1}^{\infty}b_{nk+1}r^{nk+1}}\right\}>\beta \quad (16)$$

Since $\Re e\left(-e^{i\theta}\right)\geq-\left|e^{i\theta}\right|=-1$

The above inequality (16) reduces to

$$\Re e\left\{\frac{(1-\beta)r-\sum_{n=1}^{\infty}\{(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)\}b_{nk+1}r^{nk+1}}{r-\sum_{n=1}^{\infty}b_{nk+1}r^{nk+1}}\right\}\geq 0 \quad (17)$$

Upon clearing the denominator and letting $r\rightarrow 1$ in (15) we get

$$\sum_{n=1}^{\infty}\{(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)\}b_{nk+1}\leq(1-\beta)$$

which is the result in (10). Hence the Theorem.

Corollary 2.3 If $T^k f(z)\in R_T(\lambda,\alpha,k,\beta)$ then

$$b_{nk+1}\leq\frac{(1-\beta)}{\{(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)\}}\ \forall n\geq 1$$

This result is sharp for each n for functions of the form

$$T^k f_n(z)=z-\frac{(1-\beta)}{\{(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)\}}z^{nk+1}\ \forall\ n\geq 1 \quad (18)$$

3. Distortion and Covering theorems for the function $f\in R_T(\lambda,\alpha,k,\beta)$

Theorem 3.1. If the function $T^k f(z)\in R_T(\lambda,\alpha,k,\beta)$, then

$$r-\frac{(1-\beta)}{\{(k+1)(\lambda k+1)(1+\alpha)-(\alpha+\beta)\}}r^{k+1}\leq\left|T^k f(z)\right|$$
$$\leq r+\frac{(1-\beta)}{\{(k+1)(\lambda k+1)(1+\alpha)-(\alpha+\beta)\}}r^{k+1};\ \forall\ 0<|z|=r<1 \quad (19)$$

The equality in (19) is attained for the function $T^k(f(z))$ is given by

$$T^k(f(z))=z-\frac{(1-\beta)}{\{(k+1)(\lambda k+1)(1+\alpha)-(\alpha+\beta)\}}z^{k+1} \quad (20)$$

Proof: Since $T^k f(z)\in R_T(\lambda,\alpha,k,\beta)$, from the inequality (10), we have

$$\sum_{n=1}^{\infty}\{(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)\}b_{nk+1}\leq(1-\beta)$$

It can be easily seen that

$$\{(k+1)(\lambda k+1)(1+\alpha)-(\alpha+\beta)\}\sum_{n=1}^{\infty}b_{nk+1}\leq\sum_{n=1}^{\infty}\{(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)\}b_{nk+1}$$
$$\leq(1-\beta)$$

$$\Rightarrow \sum_{n=1}^{\infty} b_{nk+1} \le \frac{(1-\beta)}{\{(k+1)(\lambda k+1)(1+\alpha)-(\alpha+\beta)\}} \tag{21}$$

Consider

$$\left|T^k(f(z))\right| = \left|z-\sum_{n=1}^{\infty} b_{nk+1} z^{nk+1}\right| \le |z| + \sum_{n=1}^{\infty} b_{nk+1}|z|^{nk+1}$$

$$\le r + \frac{(1-\beta)}{\{(k+1)(\lambda k+1)(1+\alpha)-(\alpha+\beta)\}} r^{k+1} \quad \text{(Using (21)} \tag{22}$$

This gives the right hand side of (20). Similarly

$$\left|T^k(f(z))\right| = \left|z-\sum_{n=1}^{\infty} b_{nk+1} z^{nk+1}\right| \ge |z| - \sum_{n=1}^{\infty} b_{nk+1}|z|^{nk+1}$$

$$\ge r - r^{k+1}\sum_{n=1}^{\infty} b_{nk+1}$$

$$\ge r - \frac{(1-\beta)}{\{(k+1)(\lambda k+1)(1+\alpha)-(\alpha+\beta)\}} r^{k+1} \tag{23}$$

This is the left hand side of (19). It can be easily seen that the function $T^k f(z)$ defined by (20) is the extremal function for the result in (19).

Theorem 3.2 If $T^k(f(z)) \in R_T(\lambda,\alpha,k,\beta)$, then

$$1-\frac{(k+1)(1-\beta)}{\{(k+1)(\lambda k+1)(1+\alpha)-(\alpha+\beta)\}} r^k \le \left[T^k(f(z))\right]'$$

$$\le 1+\frac{(k+1)(1-\beta)}{\{(k+1)(\lambda k+1)(1+\alpha)-(\alpha+\beta)\}} r^k ; \forall\ 0<|z|=r<1 \tag{24}$$

The equality in (24) holds true for the function $T^k(f(z))$ given by (20).

Proof: Since $T^k(f(z)) \in R_T(\lambda,\alpha,k,\beta)$, we have

$$\left|\left[T^k(f(z))\right]'\right| \le 1+\sum_{n=1}^{\infty}(nk+1) b_{nk+1}|z|^{nk}$$

$$\le 1 + r^k \sum_{n=1}^{\infty}(nk+1) b_{nk+1} \tag{25}$$

And

$$\left|\left[T^k(f(z))\right]'\right| \ge 1-\sum_{n=1}^{\infty}(nk+1) b_{nk+1}|z|^{nk}$$

$$\ge 1 - r^k \sum_{n=1}^{\infty}(nk+1) b_{nk+1} \tag{26}$$

The result in (24) holds true from (25) & (26) and using the simple consequence of(23) given by

$$\sum_{n=1}^{\infty}(nk+1) b_{nk+1} \le \frac{(1+k)(1-\beta)}{\{(k+1)(\lambda k+1)(1+\alpha)-(\alpha+\beta)\}}$$

The result is sharp for the function f given in (22).

4. Closure Theorem for the class $R_T(\lambda,\alpha,k,\beta)$

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In this section we prove that the class $R_T(\lambda,\alpha,k,\beta)$ is closed under convex linear combinations.

Theorem 4.1 If $T^k(f_0(z)) = z$ and $T^k f_n(z) = z - \frac{(1-\beta)}{\{(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)\}} z^{nk+1}; \forall\, n \ge 1$

then $T^k\{f(z)\} \in R_T(\lambda,\alpha,k,\beta)$ if and only if $T^k\{f(z)\} = \sum_{n=0}^{\infty} \mu_n T^k\{f_n(z)\}$ where $\mu_n \ge 0$ and $\sum_{n=0}^{\infty} \mu_n = 1$.

Proof: Suppose $T^k\{f(z)\} = \sum_{n=0}^{\infty} \mu_n T^k\{f_n(z)\}$ with $\mu_n \ge 0$ and $\sum_{n=0}^{\infty} \mu_n = 1$. Since

$$\sum_{n=0}^{\infty} \mu_n T^k\{f_n(z)\} = \mu_0 T^k\{f_0(z)\} + \sum_{n=1}^{\infty} \mu_n T^k\{f_n(z)\}$$

$$= \left(1 - \sum_{n=1}^{\infty} \mu_n\right) T^k\{f_0(z)\} + \sum_{n=1}^{\infty} \mu_n \left\{ z - \frac{(1-\beta)}{\{(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)\}} z^{nk+1} \right\}$$

$$= \left(1 - \sum_{n=1}^{\infty} \mu_n\right) z + \sum_{n=1}^{\infty} \mu_n \left\{ z - \frac{(1-\beta)}{\{(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)\}} z^{nk+1} \right\}$$

$$= z - \sum_{n=1}^{\infty} \mu_n \frac{(1-\beta)}{\{(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)\}} z^{nk+1}$$

Consider

$$\sum_{n=1}^{\infty} \mu_n \frac{(1-\beta)}{\{(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)\}} \times \frac{\{(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)\}}{(1-\beta)} = \sum_{n=1}^{\infty} \mu_n = 1 - \mu_0 \le 1$$

Thus the coefficients of $T^k\{f(z)\}$ satisfy the inequality (10). Hence from the Theorem (2.2) it follows that $T^k\{f(z)\} \in R_T(\lambda,\alpha,k,\beta)$.

Conversely suppose that $T^k\{f(z)\} \in R_T(\lambda,\alpha,k,\beta)$. Since

$$b_{nk+1} \le \frac{(1-\beta)}{\{(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)\}}; \forall\, n \ge 1$$

By setting $\mu_n = \frac{\{(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)\}}{(1-\beta)} b_{nk+1}; \quad n = 1,2,3,......$

and $\mu_0 = 1 - \sum_{n=1}^{\infty} \mu_n$ then $f(z) = \sum_{n=0}^{\infty} \mu_n f_n(z)$. This completes the proof of the theorem.

Theorem 4.2. The class $R_T(\lambda,\alpha,k,\beta)$ is closed under convex linear combinations.

Proof: Suppose that each of the function

$$T^k\{f_l(z)\} = z - \sum_{n=1}^{\infty} b_{nk+1} z^{nk+1} (l = 1,2)$$

is in the class $R_T(\lambda,\alpha,k,\beta)$. We need to prove that the function $H(z)$ given by

$$H(z) = \lambda_1 T^k\{f_1(z)\} + (1-\lambda_1) T^k\{f_2(z)\}; \quad (0 \le \lambda \le 1)$$

also lies in the class $R_T(\lambda,\alpha,k,\beta)$. Since

$$H(z) = z - \sum_{n=1}^{\infty} \{\lambda_1 b_{nk+1,1} + (1-\lambda_1) b_{nk+1,2}\} z^{nk+1}$$

Consider

$$\sum_{n=1}^{\infty}\{(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)\}\{\lambda_1 b_{nk+1,1}+(1-\lambda_1)b_{nk+1,2}\}$$

$$=\lambda_1\sum_{n=1}^{\infty}\{(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)\}b_{nk+1,1}+(1-\lambda_1)\sum_{n=1}^{\infty}\{(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)\}b_{nk+1,2}$$

$$\le\lambda_1(1-\beta)+(1-\lambda_1)\beta$$

$$\le(1-\beta)$$

Thus from the Theorem (2.2) $H(z)=R_T(\lambda,\alpha,k,\beta)$.Hence the class $R_T(\lambda,\alpha,k,\beta)$ is closed under convex linear combinations.

5. Radii of starlikeness, convexity and close to convexity for the functions f in the class $R_T(\lambda,\alpha,k,\beta)$.

In this section we determine radius of starlikeness, convexity and close to convexity for the function $T^k\{f(z)\}\in R_T(\lambda,\alpha,k,\beta)$.

Theorem 5.1. If c then $T^k\{f(z)\}$is starlike of order $\rho(0\le\rho<1)$ in $|z|<r_1(\lambda,\alpha,k,\beta,\rho)$ where

$$r_1(\lambda,\alpha,\kappa,\beta,\rho)=\left\{\frac{(1-\rho)(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)}{(1-\beta(nk+1-\rho))}\right\}^{\frac{1}{nk}};\quad\forall\, n\ge1$$

And the result is sharp.

Proof: Suppose $T^k\{f(z)\}\in R_T(\lambda,\alpha,k,\beta)$. It is sufficient to show that

$$\left|\frac{z[T^k\{f(z)\}]'}{T^k\{f(z)\}}-1\right|\le|1-\rho|\text{ for }0\le\rho<1,\ |z|<r_1(\lambda,\alpha,k,\beta,\rho)\qquad(27)$$

Replacing $T^k\{f(z)\}$and $z[T^k\{f(z)\}]'$ in the L.H.S of (25) with their equivalent expressions in series, we get

$$\left|\frac{\sum_{n=1}^{\infty}(nk)b_{nk+1}z^{nk}}{1-\sum_{n=1}^{\infty}b_{nk+1}z^{nk}}\right|\le\frac{\sum_{n=1}^{\infty}(nk)b_{nk+1}|z|^{nk}}{1-\sum_{n=1}^{\infty}b_{nk+1}|z|^{nk}}$$

This will be bounded by $(1-\rho)$ if

$$\sum_{n=1}^{\infty}(nk)b_{nk+1}|z|^{nk}\le(1-\rho)\left[1-\sum_{n=1}^{\infty}b_{nk+1}|z|^{nk}\right]$$

$$\sum_{n=1}^{\infty}\left\{\frac{(nk+1-\rho)}{(1-\rho)}\right\}b_{nk+1}|z|^{nk}\le1\qquad(28)$$

Since for$T^k\{f(z)\}\in R_T(\lambda,\alpha,k,\beta)$, from Theorem (2.2), we have

$$\sum_{n=1}^{\infty}\left\{\frac{(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)}{(1-\beta)}\right\}b_{nk+1}\le1$$

The condition (28) will be satisfied if

$$\frac{(nk+1-\rho)}{(1-\rho)}|z|^{nk} \leq \left\{\frac{(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)}{(1-\beta)}\right\}$$

$$|z| \leq \left\{\frac{(1-\rho)(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)}{(1-\beta)(nk+1)(nk+1-\rho)}\right\}^{\frac{1}{nk}}; \forall\ n=1,2,3,\ldots\ldots \tag{29}$$

Setting $|z| = r_1(\lambda,\alpha,k,\beta,\rho)$, the result of the theorem follows. And the result is sharp for each n for the functions $T^k\{f_n(z)\}$ given in (16).

Theorem 5.2. If $T^k\{f(z)\} \in R_T(\lambda,\alpha,k,\beta)$ then $T^k f(z)$ is close to convex of order $\rho(0 \leq \rho < 1)$ in $|z| < r_2(\lambda,\alpha,k,\beta,\rho)$ where

$$r_2(\lambda,\alpha,k,\beta,\rho) = \inf\left\{\frac{(1-\rho)(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)}{(1-\beta)(nk+1)(nk+1-\rho)}\right\}^{\frac{1}{nk}}; \forall\ n \geq 1$$

And the result is sharp.

Proof: Suppose $T^k\{f(z)\} \in R_T(\lambda,\alpha,k,\beta)$. It is sufficient to show that

$$\left|\frac{z[T^k f(z)]''}{[T^k f(z)]'}\right| \leq (1-\rho) \text{ for } [0 \leq \rho < 1,\ |z| < r_2(\lambda,\alpha,k,\beta,\rho)] \tag{30}$$

Replacing $[T^k f(z)]'$ in the L.H.S of (30) with their equivalent expressions in series then we get

$$\left|\frac{\sum_{n=1}^{\infty}(nk)(nk+1)b_{nk+1}z^{nk}}{1-\sum_{n=1}^{\infty}(nk+1)b_{nk+1}z^{nk}}\right| \leq \frac{\sum_{n=1}^{\infty}(nk)(nk+1)b_{nk+1}|z|^{nk}}{1-\sum_{n=1}^{\infty}(nk+1)b_{nk+1}|z|^{nk}}$$

This will be bounded by $(1-\rho)$ if

$$\sum_{n=1}^{\infty}(nk)(nk+1)b_{nk+1}|z|^{nk} \leq (1-\rho)\left[1-\sum_{n=1}^{\infty}(nk+1)b_{nk+1}|z|^{nk}\right]$$

Or

$$\sum_{n=1}^{\infty}\left\{\frac{(nk)(nk+1-\rho)}{(1-\rho)}\right\}b_{nk+1}|z|^{nk} \leq 1 \tag{31}$$

Since for $T^k\{f(z)\} \in R_T(\lambda,\alpha,k,\beta)$, from Theorem (2.2), we have

$$\sum_{n=1}^{\infty}\left\{\frac{\{(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)\}}{(1-\beta)}\right\}b_{nk+1} \leq 1$$

The condition will be satisfied if

$$\frac{(nk)(nk+1-\rho)}{(1-\rho)}|z|^{nk} \leq \left\{\frac{\{(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)\}}{(1-\beta)}\right\} \forall\ n \geq 1$$

$$|z| \leq \left\{\frac{(1-\rho)(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)}{(1-\beta)(nk+1)(nk+1-\rho)}\right\}^{\frac{1}{nk}} \quad \forall\ n=1,2,3,\ldots$$

Setting $|z| = r_2(\lambda,\alpha,k,\beta,\rho)$, the result of the Theorem follows. And the result is sharp for each n for the functions $T^k\{f_n(z)\}$ given in(18).

Theorem 5.3. If $T^k\{f(z)\} \in R_T(\lambda,\alpha,k,\beta)$then $T^k f(z)$ is close to convex of order $\rho(0 \le \rho < 1)$ in $|z| < r_3(\lambda,\alpha,k,\beta,\rho)$ where

$$r_3(\lambda,\alpha,k,\beta,\rho) = \inf\left\{\frac{(1-\rho)(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)}{(1-\beta)(nk+1)}\right\}^{\frac{1}{nk}};\ \forall\ n \ge 1$$

And the result is sharp.

Proof: Suppose $T^k\{f(z)\} \in R_T(\lambda,\alpha,k,\beta)$.It is sufficient to show that

$$\left|[T^k f(z)]' - 1\right| \le (1-\rho) \text{ for } [\, 0 \le \rho < 1,\ |z| < r_3(\lambda,\alpha,k,\beta,\rho)] \qquad (32)$$

Replacing $[T^k f(z)]'$ in the L.H.S of (30) with their equivalent expressions in series then we get

$$\left|1 - \sum_{n=1}^{\infty}(nk+1)b_{nk+1}z^{nk} - 1\right| \le \sum_{n=1}^{\infty}(nk+1)b_{nk+1}|z|^{nk}$$

This will be bounded by $(1-\rho)$ if

$$\sum_{n=1}^{\infty}(nk+1)b_{nk+1}|z|^{nk} \le (1-\rho)$$

$$\sum_{n=1}^{\infty}\frac{(nk+1)}{(1-\rho)}b_{nk+1}|z|^{nk} \le 1 \qquad (33)$$

Since for $T^k f(z) \in R_T(\lambda,\alpha,k,\beta)$, from Theorem (2.2), we have

$$\sum_{n=1}^{\infty}\left\{\frac{(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)}{(1-\beta)}\right\}a_n \le 1$$

The condition (33) will be satisfied if

$$\frac{(nk+1)}{(1-\rho)}|z|^{nk} \le \left\{\frac{(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)}{(1-\beta)}\right\} \quad \forall\ n \ge 1$$

$$\Rightarrow |z| \le \left\{\frac{(1-\rho)(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)}{(1-\beta)}\right\}^{\frac{1}{nk}}; \quad \forall\ n \ge 1$$

Setting $|z| = r_3(\lambda,\alpha,k,\beta,\rho)$, the result of the Theorem follows. And the result is sharp for each n for the functions $T^k[f_n(z)]$in (16).

6. Integral Operators

In this section we consider the integral operators for func tion $T^k f(z) \in R_T(\lambda,\alpha,k,\beta)$.

Theorem 6.1: If $T^k f(z) \in R_T(\lambda,\alpha,k,\beta)$ then the function $T^k[f(z)]$ defined by

$$T^k[f(z)] = \frac{1+c}{z^c}\int t^{c-1}T^k f(t)\,dt(c > -1) \qquad (34)$$

Is also in $R_T(\lambda,\alpha,k,\beta)$.

Proof: Suppose $T^k f(z) \in R_T(\lambda,\alpha,k,\beta)$, we have

$$T^k[F(z)] = z - \sum_{n=1}^{\infty} b_{nk+1}\left\{\frac{1+c}{nk+1+c}\right\} z^{nk+1}, \quad 0 < \left\{\frac{1+c}{nk+1+c}\right\} < 1$$

Consider

$$\sum_{n=1}^{\infty}\{(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)\} b_{nk+1}\left\{\frac{1+c}{nk+1+c}\right\}$$

$$\leq \sum_{n=1}^{\infty}\{(nk+1)(\lambda nk+1)(1+\alpha)-(\alpha+\beta)\} b_{nk+1}$$

$\leq (1-\beta)$ (using the inequality (10))

$$\Rightarrow \quad T^k[F(z)] \in R_T(\lambda,\alpha,k,\beta)$$

Hence the Theorem.

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