FOURTH HANKEL AND TOEPLITZ DETERMINANTS FOR STARLIKE AND CONVEX FUNCTIONS ASSOCIATED WITH COSINE FUNCTION

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Abstract : The purpose of this paper is to find the Fourth Hankel determinant for the functions belongs to the family of starlike and convex functions in connection to the cosine function. We also consider the estimation of Fekete-Szegö inequality, Zalcman conjecture and Toeplitz determinants for the functions in above classes subordinate to cosine function.

***Index Terms* - Analytic function, Convex function, Starlike function, Subordination, Function with positive real part, Hankel determinants and Toeplitz determinants.**

I. INTRODUCTION

The members f of the family $\mathcal{A}$ of holomorphic functions defined in the disc $\mathfrak{U}=\{z\in\mathbb{C}:|z|<1\}$ having Macluarins's series expansion given by

$$f(z)=z+\sum_{2\leq n}^{\infty}a_nz^n,\ z\in\mathfrak{U}, \tag{1.1}$$

are standardized by $f'(0)-1=0=f(0)$. The functions in $\mathcal{A}$ that are univalent (injective) in $\mathfrak{U}$ constitute the subclass of $\mathcal{A}$ and this subclass is represented by S. Further the two subclasses of S that describe the geometric properties of the domain onto which $\mathfrak{U}$ is mapped are respectively defined by

$$S^*=\{f\in S: Re(\tfrac{zf\prime(z)}{f(z)})>0,\ f(z)=z+\sum_{2\leq n}^{\infty}a_nz^n,\ z\in\mathfrak{U}\},$$
$$C=\{g\in S: Re(1+\tfrac{zg\prime\prime(z)}{g\prime(z)})>0,\ g(z)=z+\sum_{2\leq n}^{\infty}b_nz^n,\ z\in\mathfrak{U}\}.$$

The coefficients of functions in S^* and C are connected by the Alexander's relation [22] given by $a_n=nb_n,\forall n\in\mathbb{N}$. The well known Carathéodary class of holomorphic functions p in $\mathfrak{U}$ following $p(0)=1,0<Re(p(z)),z\in\mathfrak{U}$ is denoted by $\mathcal{P}$. The members of this class has the form $p(z)=1+\sum_{n\geq1}^{\infty}c_nz^n,z\in\mathfrak{U}$. The functions $p(z)=\frac{1+z}{1-z}$, $p(z)=\frac{1+z^2}{1-z^2}$ are in $\mathcal{P}$. Two analytic functions u and v in $\mathfrak{U}$ connected by $u(z)=v(w(z))$, for all $z\in\mathfrak{U}$, where w is a Schwarz function in $\mathfrak{U}$ satisfying $w(0)=0$ and $|w(z)|<1,z\in\mathfrak{U}$ is expressed as $u\prec v$ and read as u is subordinate to v. In addition to analyticity if v is univalent in $\mathfrak{U}$, it is evident that $u(\mathfrak{U})\subset v(\mathfrak{U})$ and $u(0)=v(0)$.

The Hankel determinant of $f\in\mathcal{A}$ for $q\geq1,n\geq1$ is designated by $H_q(n)$, defined by Pommerenke [17] as below

$$H_q(n):=\begin{vmatrix} a_n & a_{n+1} & a_{n+2}\cdots\cdots & a_{n+q-1}\\ a_{n+1} & a_{n+2} & a_{n+3}\cdots\cdots & a_{n+q}\\ a_{n+2} & a_{n+3} & a_{n+4}\cdots\cdots & a_{n+q+1}\\ -- & -- & ------- & ---\\ -- & -- & ------- & ---\\ a_{n+q-1} & a_{n+q} & a_{n+q+1}\cdots\cdots & a_{n+2q-2}\end{vmatrix}. \tag{1.2}$$

The q^{th} Hankel determinant was initially studied and developed by Noonan and Thomas [15]. For different values of q,n we get various Hankel determinants. The study of estimating sharp upper bound on $|H_q(n)|$ for functions in different subclasses of $\mathcal{A}$ attracted by many authors. For $q=2,n=1$, the determinant $|H_2(1)|$ is given by $H_2(1)=[a_3-a_2^2]$. It is the particular case of Fekete-Szegö inequality $|a_3-\nu a_2^2|$ for $\nu=1$. In case of $q=n=2$ we have $H_2(2)=a_4a_2-(a_3)^2$, the second Hankel determinant. Several researchers have studied the bound on $|H_2(2)|$. Janteng et al. [10] proved that $|H_2(2)|\leq\begin{cases}1,\text{for} f\in S^*,\\ \frac{1}{8},\text{for} f\in C.\end{cases}$. The bounds are sharp. Unfortunately, the sharp bound of $|H_2(2)|$ for $f\in S$ is still not known. Several authors like Arif et al. [1] examined $|H_2(2)|$ for various subclasses of analytic univalent and bi-univalent functions.

If we take $q=3,n=1$ in $H_q(n)$, the Hankel determinant $H_3(1)=a_3[a_4a_2-a_3^2]-a_4[a_4-a_3a_2]+a_5[a_3-a_2^2]$. By applying triangle inequality we have

$$|H_3(1)|\leq|a_3||a_4a_2-a_3^2|+|a_4||a_3a_2-a_4|+|a_5||a_3-a_2^2|. \tag{1.3}$$

All quantities on the right hand side of above expression have sharp upper bounds except $|a_2a_3-a_4|$. Babalola [3] proved that $|H_3(1)|\leq\begin{cases}16,\text{for} f\in S^*,\\ \frac{15}{24},\text{for} f\in C.\end{cases}$. Bansal et al. [4] refined the upper bound of $|H_3(1)|$ for some functions in the classes investigated by Babalola [3]. In 2017, Zaprawa [22] proved that

$|H_3(1)|\leq\begin{cases}1,\text{for} f\in S^*,\\ \frac{49}{540},\text{for} f\in C.\end{cases}$. He claimed that these bounds are still not sharp. Orhan and Zaprawa [16] obtained an upper bound to $|H_3(1)|$ for the functions in S^*,C of order α. Kowalczyk et al. [12] calculated the sharp upper bound on $|H_3(1)|$ for $f\in C$ given by $|H_3(1)|\leq\frac{4}{135}$, is a finest bound compared to the bound computed by Zaprawa [22]. Further Kwon et al. [13] estimated the bound $|H_3(1)|$ for $f \in S^*$ and the value is improved up to $\frac{8}{9}$. Ganesh et al. [7] calculated non-sharp bound on $|H_3(1)|$ for functions with respect to symmetric points associated to an exponential function. Raza and Malik [20] estimated $|H_3(1)|$ associated with Leminscate of Bernoulli.

If we take $q = 4, n = 1$ in $H_q(n)$, the fourth Hankel determinant $H_4(1)$ is given by $H_4(1) := \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \\ a_4 & a_5 & a_6 & a_7 \end{vmatrix}$.

This Hankel determinant was initially studied by Arif et al. [1] for bounded turning functions. They showed that for $f \in \mathcal{R}, |H_4(1)| \leq \frac{73757}{94500} \approx 0.78050$, where
$\mathcal{R} = \{z \in \mathfrak{U}: Re(f'(z)) > 0\}$. In 2019, Arif et al. [2] obtained bound on $|H_4(1)|$ for $f \in SL$ given by $|H_4(1)| \leq 0.0678$. In recent times, Hai-Yan Zhang et al. [8] estimated bound on $H_4(1)$ for $f \in S^*$ connected with Sine function. Muhammad Ghaffar Khan et al. [11] studied coefficient problems corresponding to bounded turning functions incorporated with Sine function. N. E. Cho and Virendra kumar [5] studied bounds on a few initial coefficients and bound on $|H_4(1)|$ for $f \in \mathcal{A}$.

The Hankel and Toeplitz determinants were closely related. Toeplitz determinants contain constant entries along the principal diagonal unlike as Hankel determinants. Thomas and Halim [21] initiated the concept of the symmetric Toeplitz determinants for f as represented in (1.1) and is defined as follows:

$$T_q(n) := \begin{vmatrix} a_n & a_{n+1} \cdots\cdots\cdots & a_{n+q-1} \\ a_{n+1} & a_n \cdots\cdots\cdots & a_{n+q-2} \\ -- & -- & --- \\ -- & -- & --- \\ a_{n+q-1} & a_{n+q-2} \cdots\cdots\cdots & a_n \end{vmatrix}, \quad (1.4)$$

where n, q are positive integers, with $a_1 = 1$. For small values of n, q the estimates of Toeplitz determinants $|T_q(n)|$ for functions in S^* and close to convex functions K have been studied in [21]. Radhika et al. [18] computed $|T_q(n)|$ for functions in $\mathcal{R}$. Md Firoz ali et al. [6] estimated $T_3(1)$ and $T_3(2)$ for f in C and R. Toeplitz determinants $T_3(1)$, $T_3(2)$, $T_2(2)$ and $T_2(3)$ for functions in M_α are estimated by Ramachandran et al. [19]. Hai-Yan Zhang et al. ([8],[9]) have been studied Toeplitz determinants $T_3(2)$ and $T_4(2)$ respectively for functions in S_s^* associated with Sine function and obtained upper bounds. Substituting $n = 1, q = 4$ and $n = 2, q = 4$ in (1.4) we have the Toeplitz determinants $T_4(1), T_4(2)$ respectively as below:

$$T_4(1) = \begin{vmatrix} 1 & a_2 & a_3 & a_4 \\ a_2 & 1 & a_2 & a_3 \\ a_3 & a_2 & 1 & a_2 \\ a_4 & a_3 & a_2 & 1 \end{vmatrix},$$

Upon simplifying we have,

$$T_4(1) = (1 - a_2^2)^2 - (a_2a_3 - a_4)^2 + (a_3^2 - a_2a_4)^2 - (a_2 - a_2a_3)^2 + 2(a_2^2 - a_3)(a_3 - a_2a_4) \quad (1.5)$$

and following [9]

$$T_4(2) = \begin{vmatrix} a_2 & a_3 & a_4 & a_5 \\ a_3 & a_2 & a_3 & a_4 \\ a_4 & a_3 & a_2 & a_3 \\ a_5 & a_4 & a_3 & a_2 \end{vmatrix},$$

$$T_4(2) = (a_2^2 - a_3^2)^2 - (a_3a_4 - a_2a_5)^2 + (a_4^2 - a_3a_5)^2 - (a_2a_3 - a_3a_4)^2 + 2(a_3^2 - a_2a_4)(a_2a_4 - a_3a_5). \quad (1.6)$$

Inspired by the work cited above we estimate fourth Hankel determinant and bounds on Toeplitz determinants $T_4(1), T_4(2)$ for starlike and convex functions related to Cosine function. Besides this we also compute Fekete-Szegö inequality and Zalcman conjecture for the functions in S^*, C subordinate to Cosine function.
Now, we define the following classes.

Definition 1.1 *An analytic function $f \in S$ is in the family S^*_{cos}, iff*

$$\frac{zf'(z)}{f(z)} \prec cos(z); \ \ z \in \mathfrak{U}.$$

Definition 1.2 *An analytic function $g \in S$ is in the family C_{cos}, iff*

$$1 + \frac{zg''(z)}{g'(z)} \prec cos(z); \ \ z \in \mathfrak{U}.$$

II. Preliminaries

Lemma 2.1 *[17] If $p(z) = \sum_{k=0}^{\infty} c_k z^k \in \mathcal{P}, \ c_0 = 1, \ z \in \mathfrak{U}$ then $|c_k| \leq 2, \forall k \in \mathbb{N}$.*

Lemma 2.2 *[2] If $p(z) = \sum_{k=0}^{\infty} c_k z^k \in \mathcal{P}, \ c_0 = 1, \ z \in \mathfrak{U}$ then for any real numbers J, K, L*

$$|Jc_1^3 - Kc_1c_2 + Lc_3| \leq 2(|J| + |K - 2J| + |J - K + L|).$$

In case, when $J = 1, K = 2, L = 1$ we have $|c_1^3 - 2c_1c_2 + c_3| \leq 2$.

Lemma 2.3 *[14] If $p(z)=\sum_{k=0}^{\infty} c_k z^k \in \mathcal{P},\ c_0=1,\ z\in \mathfrak{U}$ then for any $\eta\in\mathbb{C}$,*

$$|c_2-\eta c_1^2|\le 2\ Max\{1,|2\eta-1|\}.$$

The inequality is sharp for $p(z)=(1+z)(1-z)^{-1},\ p(z)=(1+z^2)(1-z^2)^{-1}$.

Lemma 2.4 *[22] If $p(z)=\sum_{k=0}^{\infty} c_k z^k \in \mathcal{P},\ c_0=1,\ z\in \mathfrak{U}$ then for any $\rho\in[0,1]$,*

$$|c_k-\rho c_n c_{k-n}|\le 2,$$

holds for $k,n\in\mathbb{N}, k>n$.

III. FOURTH HANKEL DETERMINANT FOR $f\in S_{cos}^*$

Theorem 3.1 *If $f\in S_{cos}^*$ is of the form (1.1) then* $|H_4(1)|\le \frac{100768}{82944}\approx 1.21489198.$ (3.1)

Proof. By the definition of the class S_{cos}^* and from the subordination it is evident that

$$\frac{zf\prime(z)}{f(z)}=cos(w(z)), \tag{3.2}$$

where

$$\frac{zf\prime(z)}{f(z)}=1+a_2z+(-a_2^2+2a_3)z^2+(3a_4-3a_2a_3+a_2^3)z^3+(4a_5+4a_2^2a_3-4a_2a_4-2a_3^2-a_2^4)z^4$$
$$+(5a_6-5a_2a_5+5a_2^2a_4+5a_2a_3^2-5a_2^3a_3+a_2^5-5a_3a_4)z^5$$
$$+(6a_7-6a_2a_6+6a_2^2a_5-6a_3a_5+12a_2a_3a_4-a_2^6-6a_2^3a_4-3a_4^2+2a_3^3-9a_2^2a_3^2+6a_2^4a_3)z^6+\cdots$$

and $w(z)$ is a Schwarz function. The functions in $\mathcal{P}$ are nicely connected with Schwarz function, we have

$$p(z)=\frac{1+w(z)}{1-w(z)}=1+c_1z+c_2z^2+c_3z^3+\cdots,$$

where $p\in\mathcal{P}$. Rewriting w in terms of p, we have

$$w(z)=\frac{c_1z}{2}+\left(\frac{c_2}{2}-\frac{c_1^2}{4}\right)z^2+\left(\frac{c_3}{2}+\frac{c_1^3}{8}-\frac{c_1c_2}{2}\right)z^3+\left(\frac{c_4}{2}-\frac{c_1c_3}{2}-\frac{c_2^2}{4}+\frac{3c_1^2c_2}{8}-\frac{c_1^4}{16}\right)z^4+\cdots.$$

Then

$$cos(w(z))$$
$$=1-\frac{c_1^2z^2}{8}+\left(\frac{c_1^3}{8}-\frac{c_1c_2}{4}\right)z^3-\frac{1}{2}\left(\frac{c_1c_3}{2}+\frac{c_2^2}{4}-\frac{3c_1^2c_2}{4}+\frac{35c_1^4}{192}\right)z^4+\left(-\frac{c_1c_4}{4}+\frac{3c_1^2c_3}{8}+\frac{3c_1c_2^2}{8}+\frac{11c_1^5}{192}-\frac{35c_1^3c_2}{96}-\frac{c_2c_3}{4}\right)z^5$$
$$+\cdots. \tag{3.3}$$

By writing series expansion of the function on the left hand side of (3.2) and comparing the like powers of $z^k, k=1,2,3,..,$ from (3.2) and (3.3), the first few coefficients are

$$a_2=0, \tag{3.4}$$
$$a_3=-\frac{c_1^2}{16}, \tag{3.5}$$
$$a_4=\frac{c_1^3}{24}-\frac{c_1c_2}{12}, \tag{3.6}$$
$$a_5=\frac{1}{4}\left(-\frac{c_1}{24}(2c_1^3-9c_1c_2+6c_3)-\frac{c_2^2}{8}\right), \tag{3.7}$$
$$a_6=\frac{1}{5}\left(\frac{c_1^2}{384}(17c_1^3-150c_1c_2+144c_3)-\frac{c_1}{4}(c_4-c_2^2)-\frac{c_2}{4}\left(c_3-\frac{c_1c_2}{2}\right)\right), \tag{3.8}$$

$$a_7=\frac{1}{6}\left(\frac{177c_1^4}{768}\left(c_2-\frac{1757}{21240}c_1^2\right)+\frac{3c_1^2}{8}\left(c_4-\frac{131}{144}c_1c_3\right)+\frac{3c_1c_2}{4}\left(c_3-\frac{395c_1c_2}{576}\right)-\frac{c_2}{4}\left(c_4-\frac{c_2^2}{2}\right)-\frac{3c_1^4}{512}-\frac{c_3^2}{8}-\frac{c_1c_5}{4}\right). \tag{3.9}$$

By taking modulus on either side of the equations (3.4)–(3.9) and applying Lemmas 2.1, 2.2, 2.3 and 2.4 respectively the bounds on these coefficients are given by

$$|a_3|\le\frac{1}{4} \tag{3.10}$$
$$|a_4|\le\frac{1}{3} \tag{3.11}$$
$$|a_5|\le\frac{11}{24} \tag{3.12}$$
$$|a_6|\le 1 \tag{3.13}$$
$$|a_7|\le\frac{453}{144} \tag{3.14}$$

Further we have

$$a_2a_4-a_3^2=-\frac{c_1^4}{256}, \tag{3.15}$$
$$a_2a_3-a_4=\frac{c_1}{12}\left(c_2-\frac{c_1^2}{2}\right), \tag{3.16}$$
$$a_3-a_2^2=-\frac{c_1^2}{16}, \tag{3.17}$$
$$a_2a_4+2a_3^2=\frac{c_1^4}{128}. \tag{3.18}$$

By applying Lemma 2.1 on (3.15), (3.17), (3.18) and Lemma 2.4 on (3.16), we have

$$|a_2a_4-a_3^2|\le\frac{1}{16},$$
$$|a_2a_3-a_4|\le\frac{1}{3}, \tag{3.19}$$
$$|a_3-a_2^2|\le\frac{1}{4},$$

$$|a_2a_4 + 2a_3^2| \leq \frac{1}{8}.$$

From (1.3) and (3.19) along with (3.10), (3.11), (3.12) we have $|H_3(1)| \leq \frac{79}{576}$. Following the representation given in [11] the bound on fourth Hankel determinant is given by

$$|H_4(1)| \leq |a_7||H_3(1)| + 2|a_4||a_6||a_2a_4 - a_3^2| + 2|a_5||a_6||a_2a_3 - a_4| + |a_6|^2|a_3 - a_2^2| + |a_5|^2|a_2a_4 - a_3^2| + |a_5|^2|a_2a_4 + 2a_3^2| + |a_5|^3 + |a_4|^4 + 3|a_3||a_4|^2|a_5| \quad (3.20)$$

By employing the respective bounds on the right hand side of (3.20) from (3.10)–(3.13) along with (3.19), we have the desired result.

Theorem 3.2 *If $f \in S_{cos}^*$, then $|a_3 - \nu a_2^2| \leq \frac{1}{4}$* for any $\nu \in \mathbb{C}$.

Proof. From (3.4) and (3.5), we have for any $\nu \in \mathbb{C}$,

$$|a_3 - \nu a_2^2| = |a_3| = \left|\frac{c_1^2}{16}\right|.$$

By Lemma 2.1 the result follows.

IV. ZALCMAN CONJECTURE FOR $f \in S_{cos}^*$

In this section we consider Zalcman conjecture for the case of n=2,3,4.

Theorem 4.1 *If $f \in S_{cos}^*$ is of the form (1.1) then*

$$|a_2^2 - a_3| \leq \frac{1}{4},$$
$$|a_3^2 - a_5| \leq \frac{41}{96}, \quad (4.1)$$
$$|a_4^2 - a_7| \leq \frac{81}{18}.$$

Proof. From Theorem 3.2, it is evident that $|a_2^2 - a_3| \leq \frac{1}{4}$, for $\nu = 1$.

From (3.5) and (3.7), we have

$$a_3^2 - a_5 = (-\frac{c_1^2}{16})^2 - \frac{1}{4}(-\frac{c_1}{24}(2c_1^3 - 9c_1c_2 + 6c_3) - \frac{c_2^2}{8}) = \frac{c_2^2}{32} + \frac{c_1}{4}(\frac{19c_1^3 - 72c_1c_2 + 48c_3}{192})$$

Taking modulus and employing Lemmas 2.1, 2.2 we get

$$|a_3^2 - a_5| \leq \frac{41}{96}.$$

Again from (3.6), (3.9) we have

$$a_4^2 - a_7 = \frac{1}{6}(\frac{2717c_1^6}{92160} - \frac{209c_1^4c_2}{768} + \frac{427c_1^2c_2^2}{768} + \frac{131c_1^3c_3}{384} + \frac{c_3^2}{8} - \frac{3c_1c_2c_3}{4} + \frac{c_1c_5}{4} - \frac{3c_1^2c_4}{8} + \frac{c_2c_4}{4} - \frac{c_2^3}{8})$$
$$= \frac{1}{6}(-\frac{209c_1^4}{768}(c_2 - \frac{2717c_1^2}{25080}) - \frac{3c_1c_2}{4}(c_3 - \frac{427c_1c_2}{576}) + \frac{c_1}{4}(c_5 - \frac{c_2c_3}{2}) + \frac{c_3}{384}(131c_1^3 + 48c_1c_2 + 48c_3) + \frac{c_2}{4}(c_4 - \frac{c_2^2}{2}) - \frac{3c_1^2c_4}{8})$$

Upon taking modulus and applying Lemma 2.1, 2.2, 2.4 we get

$$|a_4^2 - a_7| \leq \frac{81}{18}.$$

Hence Zalcman conjecture holds for $n = 2,3$ and 4.

V. FOURTH HANKEL DETERMINANT FOR $g \in C_{cos}$

Theorem 5.1 *If $g(z) = z + \sum_{2\leq n}^{\infty} b_nz^n \in C_{cos}$, then* $|H_4(1)| \leq \frac{2834946}{217728000} \approxeq 0.0130205853 \quad (5.1)$

Proof. From Theorem 3.1 and the Alexander's relation $a_n = nb_n$ we get the coefficients of g as

$$b_2 = 0, \quad (5.2)$$
$$b_3 = -\frac{c_1^2}{48}, \quad (5.3)$$
$$b_4 = \frac{1}{4}(\frac{c_1^3}{24} - \frac{c_1c_2}{12}), \quad (5.4)$$
$$b_5 = \frac{1}{20}(-\frac{c_1^4}{12} + \frac{3c_1^2c_2}{8} - \frac{c_1c_3}{4} - \frac{c_2^2}{8}), \quad (5.5)$$
$$b_6 = \frac{1}{30}\left(\frac{17c_1^5}{384} - \frac{75c_1^3c_2}{192} + \frac{3c_1^2c_3}{8} - \frac{c_1c_4}{4} + \frac{c_1c_2^2}{4} - \frac{c_2c_3}{4} + \frac{c_1c_2^2}{8}\right), \quad (5.6)$$
$$b_7 = \frac{1}{42}(\frac{177c_1^4c_2}{768} - \frac{1757}{92160}c_1^6 + \frac{3c_1^2c_4}{8} - \frac{131}{384}c_1^3c_3 + \frac{3c_1c_2c_3}{4} - \frac{395c_1^2c_2^2}{768} - \frac{c_2c_4}{4} + \frac{c_2^3}{8} - \frac{c_3^2}{8} - \frac{c_1c_5}{4}). \quad (5.7)$$

The bounds on these coefficients is readily given from (5.3)–(5.7) and we have

$$|b_3| \leq \frac{1}{12} \quad (5.8)$$
$$|b_4| \leq \frac{1}{12} \quad (5.9)$$
$$|b_5| \leq \frac{11}{120} \quad (5.10)$$
$$|b_6| \leq \frac{1}{6} \quad (5.11)$$

$$|b_7| \leq \frac{453}{1008} \tag{5.12}$$

Further we have

$$b_2b_4 - b_3^2 = -\frac{c_1^4}{2304}, \tag{5.13}$$

$$b_2b_3 - b_4 = \frac{c_1}{48}(c_2 - \frac{c_1^2}{2}), \tag{5.14}$$

$$b_3 - b_2^2 = -\frac{c_1^2}{48}, \tag{5.15}$$

$$b_2b_4 + 2b_3^2 = \frac{c_1^4}{1152}. \tag{5.16}$$

By applying Lemma 2.1 on (5.13), (5.15), (5.16) and Lemma 2.4 on (5.14), we have

$$\begin{aligned} |b_2b_4 - b_3^2| &\leq \frac{1}{144}, \\ |b_2b_3 - b_4| &\leq \frac{1}{12}, \\ |b_3 - b_2^2| &\leq \frac{1}{12}, \\ |b_2b_4 + 2b_3^2| &\leq \frac{1}{72}. \end{aligned} \tag{5.17}$$

From (1.3) and (5.17) along with (5.8), (5.9), (5.10) we have $|H_3(1)| \leq \frac{131}{8640}$. Following the representation given in [11] the bound on fourth Hankel determinant is given by

$$|H_4(1)| \leq |b_7||H_3(1)| + 2|b_4||b_6||b_2b_4 - b_3^2| + 2|b_5||b_6||b_2b_3 - b_4| + |b_6|^2|b_3 - b_2^2| + |b_5|^2|b_2b_4 - b_3^2| + |b_5|^2|b_2b_4 + 2b_3^2| + |b_5|^3 + |b_4|^4 + 3|b_3||b_4|^2|b_5| \tag{5.18}$$

By employing the respective bounds on the right hand side of (5.18) from (5.8)–(5.11) along with (5.17), we have the desired result.

Theorem 5.2 *If $g \in C_{cos}$, then*

$$|b_3 - \varrho b_2^2| \leq \frac{1}{12}, \tag{5.19}$$

for any $\varrho \in \mathbb{C}$.

Proof. From (5.2) and (5.3), we have for any $\varrho \in \mathbb{C}$,

$$|b_3 - \varrho b_2^2| = |b_3| = \left|\frac{c_1^2}{48}\right|.$$

By Lemma 2.1 the result follows.

VI. ZALCMAN CONJECTURE FOR $g \in C_{cos}$

In this section we consider Zalcman conjecture for the case of n=2,3,4.

Theorem 6.1 *If $g \in C_{cos}$ is of the form (1.1) then*

$$\begin{aligned} |b_2^2 - b_3| &\leq \frac{1}{12}, \\ |b_3^2 - b_5| &\leq \frac{127}{1440}, \\ |b_4^2 - b_7| &\leq \frac{2975}{4032}. \end{aligned} \tag{6.1}$$

Proof. From Theorem 5.2, it is evident that $|b_2^2 - b_3| \leq 0.0833$, for $\varrho = 1$.
From (5.3) and (5.5), we have

$$\begin{aligned} b_3^2 - b_5 &= (-\frac{c_1^2}{48})^2 - \frac{1}{20}(-\frac{c_1^4}{12} + \frac{3c_1^2c_2}{8} - \frac{c_1c_3}{4} - \frac{c_2^2}{8}) \\ &= \frac{c_2^2}{160} + \frac{c_1}{80}(\frac{53c_1^3 - 216c_1c_2 + 144c_3}{144}) \end{aligned}$$

Taking modulus and employing Lemma 2.1, 2.2 we get

$$|b_3^2 - b_5| \leq \frac{127}{1440} = 0.0881.$$

Again from (5.4), (5.7) we have

$$b_4^2 - b_7 = \frac{1}{42}(\frac{2177c_1^6}{92160} - \frac{573c_1^4c_2}{2304} + \frac{1227c_1^2c_2^2}{2304} - \frac{131c_1^3c_3}{384} - \frac{c_3^2}{8} + \frac{3c_1c_2c_3}{4} - \frac{c_1c_5}{4} + \frac{3c_1^2c_4}{8} - \frac{c_2c_4}{4} + \frac{c_2^3}{8})$$

$$= \frac{1}{42}(-\frac{573c_1^4}{2304}(c_2 - \frac{2177c_1^2}{22920}) - \frac{c_1}{4}(c_5 - \frac{c_2c_3}{2}) - \frac{c_3}{384}(131c_1^3 - 288c_1c_2 + 48c_3) - \frac{c_2}{4}(c_4 - \frac{c_2^2}{2}) + \frac{1227c_1^2c_2}{2304} + \frac{3c_1^2c_4}{8} - \frac{c_1c_2c_2}{8})$$

Upon taking modulus and applying Lemma 2.1, 2.2, 2.4 we get

$$|b_4^2 - b_7| \leq \frac{2975}{4032}.$$

Hence Zalcman conjecture holds for $n = 2,3and4$.

VII. BOUNDS ON TOEPLITZ DETERMINANT $T_4(1), T_4(2)$ for $f \in S^*_{cos}$

Theorem 7.1 *If $f \in S^*_{cos}$ and $f(z) = z + \sum_{2\le n}^{\infty} a_n z^n$, then* $|T_4(1)| \le \frac{2857}{2304} \approxeq 1.24001736.$ (7.1)

Proof. From (1.5), substituting the value of $a_2 = 0$, taking modulus on either side of (1.5) and applying triangle inequality we have

$$|T_4(1)| \le 1 + |a_4|^2 + |a_3|^4 + 2|a_3|^2$$
$$= 1 + (\tfrac{1}{3})^2 + (\tfrac{1}{4})^4 + 2(\tfrac{1}{4})^2$$
$$= \frac{2857}{2304} \approxeq 1.24001736.$$

Theorem 7.2 *If $f \in S^*_{cos}$ and $f(z) = z + \sum_{2\le n}^{\infty} a_n z^n$, then* $|T_4(2)| \le \frac{3753}{82944} \approxeq 0.0452473958.$ (7.2)

Proof. From (3.5), (3.6), (3.7) we have

$$a_4^2 - a_3a_5 = (\frac{c_1^3}{24} - \frac{c_1c_2}{12})^2 - (-\frac{c_1^2}{16})(\frac{1}{4}(-\frac{c_1}{24}(2c_1^3 - 9c_1c_2 + 6c_3) - \frac{c_2^2}{8}))$$
$$= \frac{c_1^2}{16}[-\frac{5c_1^2}{288}(c_2 - \frac{2}{5}c_1^2) + \frac{23}{288}(c_4 - \frac{18}{23}c_1c_3) - \frac{23}{288}(c_4 - c_2^2)] \quad (7.3)$$

Taking modulus and applying lemmas 2.3, 2.4 along with the triangle inequality we obtain

$$|a_4^2 - a_3a_5| \le \frac{4}{16}(\frac{5(8)}{288} + \frac{23(4)}{288}) = \frac{33}{288} = 0.1145 \quad (7.4)$$

From (1.6), substituting the value of $a_2 = 0$, taking modulus on either side of (1.6) and applying triangle inequality we have

$$|T_4(2)| \le |a_3|^4 + 2|a_3|^3|a_5| + 2|a_3|^2|a_4|^2 + |a_4^2 - a_3a_5|^2$$
$$= (\frac{1}{4})^4 + 2(\frac{1}{4})^3\frac{11}{24} + 2(\frac{1}{4})^2(\frac{1}{3})^2 + (\frac{33}{288})^2$$
$$= \frac{3753}{82944} \approxeq 0.0452473958 \quad (7.5)$$

VIII. BOUNDS ON TOEPLITZ DETERMINANT $T_4(1), T_4(2)$ FOR $g \in C_{cos}$

The results of the following theorems are analogous to the theorems discussed under section 7, hence the details are omitted.

Theorem 8.1 *If $g \in C_{cos}$ is of the form (1.1) then* $|T_4(1)| \le \frac{21169}{20736} \approxeq 1.02088156$ (8.1)

Proof. We have

$$|T_4(1)| \le 1 + |b_4|^2 + |b_3|^4 + 2|b_3|^2$$
$$= 1 + (\frac{1}{12})^2 + (\frac{1}{12})^4 + 2(\frac{1}{12})^2$$
$$= \frac{21169}{20736} \approxeq 1.02088156$$

Theorem 8.2 *If $g \in C_{cos}$ is of the form (1.1) then* $|T_4(2)| \le \frac{601}{2073600} \approxeq 0.000289834105.$ (8.2)

Proof. Consider

$$b_4^2 - b_3b_5 = \frac{1}{16}(\frac{c_1^3}{24} - \frac{c_1c_2}{12})^2 - (-\frac{c_1^2}{48})(\frac{1}{20})[-\frac{c_1^4}{12} + (\frac{3c_1^2c_2}{8}) - (\frac{c_1c_3}{4}) - (\frac{c_2^2}{8})]$$
$$= \frac{c_1^2}{64}[-\frac{c_1^2}{360}(c_2 - \frac{1}{2}c_1^2) + \frac{7}{360}(c_4 - \frac{6}{7}c_1c_3) - \frac{7}{360}(c_4 - c_2^2)] \quad (8.3)$$

Taking modulus and applying lemmas 2.3, 2.4 along with the triangle inequality we obtain

$$|b_4^2 - b_3b_5| \le \frac{4}{64}(\frac{8}{360} + \frac{7(2)}{360}) + \frac{7(2)}{360}$$
$$= \frac{1}{160} \approxeq 0.00625 \quad (8.4)$$

From (1.6), substituting the value of $b_2 = 0$, taking modulus on either side of (1.6) and applying triangle inequality we have

$$|T_4(2)| \le |b_3|^4 + 2|b_3|^3|b_5| + 2|b_3|^2|b_4|^2 + |b_4^2 - b_3b_5|^2$$
$$= (\frac{1}{12})^4 + 2(\frac{1}{12})^3\frac{11}{120} + 2(\frac{1}{12})^2(\frac{1}{12})^2 + (\frac{1}{160})^2$$
$$= \frac{601}{2073600} \approxeq 0.000289834105 \quad (8.5)$$

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