## Chapter One

## BASIC MATHEMATICAL TOOLS

As the reader will see, the study of the time value of money involves substantial use of variables and numbers that are raised to a power. The power to which a variable is to be raised is called an exponent. For instance, the expression $4^{3}$ means four to the third power or $4 \times 4 \times 4=$ 64. In general, when we say $Y^{n}$ we mean multiply $Y$ by itself $n$ number of times.

Because of the extended use of exponents, we will briefly review the rules for dealing with powers.

Rule 1: $Y^{0}=1$
Rule 2: $Y^{m} \times Y^{n}=Y^{m+n}$

Rule 3: $Y^{m} / Y^{n}=Y^{m-n}$

Rule 4: $\left(Y^{m}\right)^{n}=\mathrm{Y}^{m n}$

Rule 5: $Y^{1 / n}=\sqrt[n]{Y}$

Rule 6: $Y^{n}=1 / Y^{n}$

Any number to the zero power is equal to one.
The product of a common number or variable with different exponents is just that number with a power equal to the sum of the exponents. This rule is also known as the product rule.

The quotient of a common number or variable with different exponents is just that number with a power equal to the difference of the exponents. This rule is also called the quotient rule.

A variable or number with an exponent that is raised to another power is equal to that number with a power equal to the product of the exponents. This rule also carries the name power rule.

A variable or number that has an exponent of the form $1 / n$ is just the $n^{\text {th }}$ root of that number. For instance, $36^{1 / 2}$ is equivalent to the square root of 36 or $\sqrt[2]{36}=\sqrt{36}=6$.

A number or variable raised to a negative power is just the reciprocal of that number raised to the positive power.

This last rule will come in very handy when working problems in the text. Few, if any, calculators will take the negative root of a number. However, by knowing that $5^{-3}$ is the same as $1 / 53$ we can easily use a calculator to find that $5^{3}=125$ and $1 / 125$ equals 0.008 .

## INTRODUCTION TO GEOMETRIC SERIES

A mathematical series is the sum of a sequence of real numbers. A series can be finite, with limited number of terms, or infinite, with unlimited number of terms. For a finite series, let $n$ be the number of terms in a series and $a_{\mathrm{i}}$ be the $\mathrm{i}^{\text {th }}$ term of the series, then a finite series can be expressed as

$$
\begin{aligned}
S_{n} & =a_{1}+a_{2}+\ldots \ldots+a_{n-1}+a_{n} \\
& =\sum_{i=1}^{n} a_{i}
\end{aligned}
$$

Consider the following example: obtain the sum of

$$
S=1+2+2^{2}+2^{3}+2^{4}
$$

By DIRECT METHOD, we have

$$
S=1+2+4+8+16=31
$$

However, consider the following:

$$
\begin{equation*}
S=1+2+2^{2}+2^{3}+2^{4} \tag{I}
\end{equation*}
$$

Multiplying both sides by 2 , we get

$$
\begin{equation*}
2 S=2+2^{2}+2^{3}+2^{4}+2^{5} \tag{II}
\end{equation*}
$$

Now we subtract equation (II) - (I), we get

$$
S=2^{5}-1=31
$$

What is unique about the above series?
For uniqueness, consider the following:
You pick any two consecutive terms, let us say $3^{\text {rd }}$ and $4^{\text {th }}$, the third term is $2^{2}$ and the fourth term is $2^{3}$. Define the ratio,

$$
R=\frac{\text { succeedingTerm }}{\text { precedingTerm }}=\frac{2^{3}}{2^{2}}=2
$$

You can see that any succeeding term over preceding term remains constant and is equal to 2 .
Now, we introduce the following definition
"In any series if the ratio of succeeding term over preceding term remains constant, then the series is known as geometric series and the ratio is known as COMMON RATIO."

With the help of the above argument, we can define a geometric series as follows
A series is said to be geometric if the ratio of each succeeding term to the preceding term always remains a constant. This constant is called the common ratio.

For example, a series such as

$$
a, a r, a r^{2} a r^{3}, \ldots, a r^{\mathrm{n}}, a r^{\mathrm{n}+1}, \ldots
$$

where the ratio

$$
\frac{a r^{\mathrm{i}+1}}{a r^{i}}=r
$$

remains constant for any two successive terms is a geometric series. If the number of terms in the series is fixed, for example, say ten or twenty, then the series is finite. If there are an infinite number of terms, the series is called an infinite series.

A series is convergent if its sum equals a finite real number. In a finite series, an expression can always be obtained for the sum of the series,

$$
S=a+a r+a r^{2}+\ldots+a r^{n-1}
$$

regardless of whether $r$ is greater or less than one (in an infinite series the thing will change see below ). An infinite series will only be convergent if the common ratio is less than one. Simple formulas for the sum of each series follow.

## Sum of a Finite Geometric Series

The sum of a finite series, $S$, can be obtained as follows:

$$
\begin{equation*}
S=a+a r+a r^{2} .+\ldots+a r^{\mathrm{n}-1} \tag{1.1}
\end{equation*}
$$

Multiplying both sides of this expression by r ,

$$
\begin{equation*}
r S=a r+a r^{2}+a r^{3}+\ldots+a r^{\mathrm{n}} \tag{1.2}
\end{equation*}
$$

Subtracting equation (1.1) from (1.2), and canceling terms, we get

$$
(r-1) S=a r^{\mathrm{n}}-a
$$

or

$$
\begin{equation*}
S=\frac{a r^{n}-a}{r-1} \tag{1.3}
\end{equation*}
$$

Thus we have the following result

The sum of a finite geometric series with common ratio $r$ is given by

$$
S=\frac{a r^{n}-a}{r-1} \text { or } \frac{a-a r^{n}}{1-r} \text { or } \frac{a\left(1-r^{n}\right)}{1-r}
$$

Note that this formula fails when $r=1$. In fact, when $r=1$

$$
\begin{align*}
S & =a+a+a+\text { to } n \text { terms } \\
& =n a \tag{1.4}
\end{align*}
$$

Thus, one does not need any algebraic formula to obtain this sum.
In the above series, the number of terms is finite. What happens when the number of terms is infinite?

## Sum of an Infinite Geometric Series

Consider the following:
We have the infinite series given by

$$
\begin{equation*}
S=a+a r+a r^{2}+\ldots \text { up to } \infty . \tag{1.5}
\end{equation*}
$$

where $r$ is assumed to be less than one.
The common ratio is ' $r$ '. If ' $r=1$ ', then

$$
S=a+a+a+\ldots \text { up to } \infty .
$$

Adding any constant up to infinity will always be infinity. Furthermore, if $r>1$, then what happens?

Every succeeding term will be greater than the preceding term.

Hence, the value of the sum $S$ will again be infinity. Therefore, it is obvious that the sum of the infinite geometric series can exist only if the common ratio $r<1$

Multiplying both sides by $r$, as in the previous case equation 1.5 , we obtain

$$
\begin{equation*}
r S=a r+a r^{2}+a r^{3}+\ldots \text { up to } \infty . \tag{1.6}
\end{equation*}
$$

Subtracting equation (1.5) from (1.6) we get

$$
(r-1) S=-a
$$

or

$$
\begin{equation*}
s=\frac{a}{1-r} \tag{1.7}
\end{equation*}
$$

Thus we have the following result

The sum of an infinite geometric series with common ratio ' $r$ ' is given by

$$
\begin{array}{rlrl}
S & =\frac{a}{1-r} & \text { for } r<1 \\
& =\infty & & \text { for } r \geq 1
\end{array}
$$

The simple examples that follow illustrate the usage of these formulas.
Example 1.1: Obtain the sum of

$$
S=1+2+2^{2}+2^{3}+2^{4} .
$$

We can obtain the sum directly as

$$
S=1+2+4+8+16=31 .
$$

In order to apply formula (1.3), note that

$$
a=1 \text {, and the common ratio is found as }
$$

$$
r=\frac{2}{1}=\frac{2^{2}}{2}=\frac{2^{3}}{2^{2}}=\frac{2^{4}}{2^{3}}=2, \text { a constant }
$$

The total number of terms, $n$, is 5 . Thus

$$
S=\frac{a\left(1-r^{n}\right)}{1-r}=\frac{1\left(1-2^{5}\right)}{1-2}=\frac{1-32}{-1}=31
$$

which is the same result if the sum is calculated directly.

In order to obtain the sum using formula (1.3), note that the sum can be written as

$$
S=2+2 \times 2+2 \times 2^{2}+2 \times 2^{3}
$$

with

$$
a=2, \text { a common ratio of } 2, \text { and } n=4 \text {. }
$$

Therefore,

$$
S=\frac{a\left(r^{n}-1\right)}{r-1}=\frac{2\left(2^{4}-1\right)}{2-1}=30
$$

Example 1.2: Obtain the sum of

$$
\begin{gathered}
S=\frac{1}{1.06}+\frac{1}{(1.06)^{2}}+\frac{1}{(1.06)^{3}+\ldots+}+\frac{1}{(1.06)^{20}} \\
S=\frac{1}{1.06}+\frac{1}{1.06} \times \frac{1}{1.06}+\frac{1}{1.06} \times \frac{1}{(1.06)^{2}}+\ldots+\frac{1}{1.06} \times \frac{1}{(1.06)^{19}}
\end{gathered}
$$

Thus

$$
\begin{gathered}
a=\frac{1}{1.06} \\
r=\frac{1}{1.06} \\
n=20 \\
S=\frac{\frac{1}{1.06}\left[1-\frac{1}{(1.06)^{20}}\right]}{1-\frac{1}{1.06}}=11.4699
\end{gathered}
$$

Example 1.3: Obtain the sum of

$$
S=\frac{10}{1.06}+\frac{10}{(1.06)^{2}}+\frac{10}{(1.06)^{3}}+\cdots \text { up to } \infty
$$

We can rewrite this as

$$
S=\frac{10}{1.06}+\frac{10}{(1.06)} \times \frac{1}{(1.06)}+\frac{10}{(1.06)} \times \frac{1}{(1.06)^{2}}+\cdots \text { up to } \infty
$$

with

$$
\begin{aligned}
& a=\frac{10}{1.06} \text { and } \\
& r=\frac{1}{1.06}<1 .
\end{aligned}
$$

Thus applying formula (1.7), we have

$$
\begin{aligned}
S & =\frac{a}{1-r}=\frac{\frac{10}{1.06}}{1-\frac{1}{1.06}}=10\left[\frac{\frac{1}{1.06}}{1-\frac{1}{1.06}}\right]=10\left[\frac{1}{1.06} \times \frac{1.06}{.06}\right] \\
& =\frac{10}{.06}=166.67
\end{aligned}
$$

Example 1.4: Obtain the sum of

$$
S=\frac{d_{1}}{(1+k)}+\frac{d_{1}(1+g)}{(1+k)^{2}}+\frac{d_{1}(1+g)^{2}}{(1+k)^{3}}+\cdots \text { up to } \infty
$$

where $k>g, d_{1}>0$.

$$
a=\frac{d_{1}}{1+k}
$$

Rewriting this series, we obtain

$$
S=\frac{d_{1}}{1+k}+\frac{d_{1}}{1+k} \times \frac{1+g}{1+k}+\frac{d_{1}}{1+k} \times\left(\frac{1+g}{1+k}\right)^{2}+\frac{d_{1}}{1+k} \times\left(\frac{1+g}{1+k}\right)^{3}+\cdots
$$

It is easy to see that the common ration $r=\frac{1+g}{1+k}<1$ since $k>g$. Hence, by formula (1.7),

$$
S=\frac{a}{1-r}=\frac{\frac{d_{1}}{1+k}}{1-\left(\frac{1+g}{1+k}\right)}=\frac{d_{1}}{1+k} \times \frac{1+k}{k-g}=\frac{d_{1}}{k-g} .
$$

People familiar with basic financial management will recognize that this is the famous Gordon's equity valuation formula, where $d_{1}=$ dividend to be paid next period, $k=$ cost of equity capital, and $g=$ earnings growth rate.

To use this equity valuation formula suppose the SBP Corporation intends to pay annual dividends of $\$ 1$ next year, and the earnings of SBP are growing at an annual rate of 6 percent. The cost of capital to the SBP Corporation is assumed to be 8 percent. If these values are expected to continue into the future, how much would a share of stock in the SBP Corporation be worth? Applying the above equity valuation formula, we have

$$
S=\frac{d_{1}}{k-g}=\frac{\$ 1}{.08-.06}=\$ 50
$$

that is, a share of stock is worth $\$ 50$.
Formulas such as these are used extensively in the development of financial models and, as can be seen from the examples, can simplify greatly the tasks at hand.

## INTRODUCTION TO ARITHMETIC SERIES

An arithmetic series is a mathematical series that the difference of any two successive members of the sequence is a constant. For instance, the sequence $3,5,7,9,11,13 \ldots$ is an arithmetic progression with common difference 2 . The constant difference is called common difference, denoted by $d$. Mathematically,

$$
\begin{equation*}
a_{i}-a_{i-1}=d, \text { for all } i \tag{1.8}
\end{equation*}
$$

Applying equation (1.8) recursively, the terms in an arithmetic series can expressed as follows:

$$
\begin{aligned}
& a_{1} \\
& a_{2}=a_{1}+d \\
& a_{3}=a_{2}+d=a_{1}+2 d \\
& \vdots \\
& a_{i}=a_{i-1}+d=a_{1}+(i-1) d \\
& \vdots \\
& a_{n}=a_{n-1}+d=a_{1}+(n-1) d
\end{aligned}
$$

For an arithmetic series with $n$ terms, using the $a_{i}$ 's computed above, the value (sum) of the series can be expressed as

$$
\begin{equation*}
S_{n}=a_{1}+\left(a_{1}+d\right)+\left(a_{1}+2 d\right)+\cdots+\left[a_{1}+(n-2) d\right]+\left[a_{1}+(n-1) d\right] \tag{1.9}
\end{equation*}
$$

Similarly, the right hand side of equation (1.9) can be written inversely as

$$
\begin{equation*}
S_{n}=\left[a_{1}+(n-1) d\right]+\left[a_{1}+(n-2) d\right]+\cdots+\left(a_{1}+2 d\right)+\left(a_{1}+d\right)+a_{1} \tag{1.10}
\end{equation*}
$$

Add both sides of equations (1.9) and (1.10). All terms involving $d$ cancel, and so we are left with:

$$
2 S_{n}=n \cdot\left[2 a_{1}+(n-1) d\right]
$$

Rearranging and remembering that $a_{n}=a_{1}+(n-1) d$, we get:

$$
\begin{equation*}
S_{n}=\frac{n\left[2 a_{1}+(n-1) d\right]}{2}=\frac{n\left(a_{1}+a_{n}\right)}{2} \tag{1.11}
\end{equation*}
$$

Intuitively, this formula can be derived by realizing that the sum of the first and last terms in the series is the same as the sum of the second and second to last terms, and so forth, and that there are roughly $n / 2$ such sums in the series. Another way to look at this is that the value of the arithmetic series is the number of terms in the series times the average value of the terms. The average must be $\left(a_{1}+a_{n}\right) / 2$, since the values appear evenly spaced out around this point on the real number line.

Example 1.5: Find the value of the following series: $S=1+2+3+4+\cdots+98+99+100$
Answer: Note that $S$ is an arithmetic series with common difference $d=1$ and we have $n=100$, $a_{1}=1$ and $a_{100}=100$. Using equation (1.11),

$$
S_{100}=\frac{100(1+100)}{2}=5,050
$$

Example 1.6: Find the value of the following series: $S=17+21+25+29+\cdots+245+249$
$\underline{\text { Answer: }}$ Note that S is an arithmetic series with common difference $\mathrm{d}=4$ and we have $a_{1}=17$ and $a_{n}=249$, but we do not know the value of $n$. Using the fact that $a_{n}=a_{1}+(n-1) d$, we get:

$$
249=17+(n-1) \times 4
$$

Solving the above equation for $n$ we obtain $n=59$. Using equation (1.11),

$$
S_{59}=\frac{59(17+249)}{2}=7,847
$$

## THE MEANING OF THE NUMBER e

Consider the expression

$$
\left(1+\frac{1}{m}\right)^{m}
$$

and let $m$ take different values (e.g., $1,000,100,000,100,000,000$, etc.). The values of the expressions for the different values of $m$ are as follows.

For, $m=1,000$

$$
\left(1+\frac{1}{1,000}\right)^{1,000}=2.716923932
$$

For, $m=100,000$

$$
\left(1+\frac{1}{100,000}\right)^{100,000}=2.718268287
$$

For, $m=100,000,000$

$$
\left(1+\frac{1}{100,000,000}\right)^{100,000,000}=2.718281828
$$

Thus, in general, we can write

$$
\begin{equation*}
\operatorname{Lim}_{m \rightarrow \infty}\left(1+\frac{1}{m}\right)^{m}=2.71828183=e \tag{1.12}
\end{equation*}
$$

This limit value yields the irrational number known as $e$.
Now consider $m$ to be the number of periods during the year that $\$ 1$ is compounded when the interest rate is 100 percent. Then when $m=\infty$, the case of continuous compounding, that dollar would be worth approximately $\$ 2.72$ at the end of a year. Further, if we use any interest rate, a, instead of 100 percent, it can be easily shown that,

$$
\begin{equation*}
\operatorname{Lim}_{n \rightarrow \infty}\left(1+\frac{a}{n}\right)^{n}=e^{a} \tag{1.13}
\end{equation*}
$$

where $a$ is any constant. Thus $\$ 10$ invested today at 8 percent would be worth $(10) e^{(.08)(1)}=\$ 10.83$ at the end of a year if the dollar were to be compounded continuously.

For those of you who are some knowledge of calculus, we can mathematically prove equation (1.13).

## EXPONENTIAL FUNCTIONS AND LOGARITHMIC FUNCTIONS

Both exponential function and its inverse, namely the logarithmic function, are widely used in the mathematics of time value of money. The basic exponential function is defined by

$$
\begin{equation*}
y=f(x)=b^{x} \tag{1.14}
\end{equation*}
$$

where $b$, a real number, is the base such that $b>0$ and b not equal to 1 . The domain of $f$ is the set of all real numbers.

## Example:

a. $y=2^{x}$
b. $y=4^{x}$
c. $y=0.4^{x}$
d. $y=1.1^{x}$

The logarithmic function is the mathematical operation that is the inverse of exponential function. The inverse of the exponential function in expression (1.14) is

$$
\begin{equation*}
x=\log _{b}(y) \tag{1.15}
\end{equation*}
$$

The domain of logarithm is the set of positive real numbers, that is, $y>0$. Note that expression reads as: $x$ equals to $\log$ to the base $b$ of $y$. Logarithms are useful in solving equations in which exponents are unknown.

The most widely used bases for logarithms are 10 , the mathematical constant $\mathrm{e} \approx 2.71828$ and 2 . When "log" is written without a base ( $b$ missing from $\log _{b}$ ), the intent can usually be determined from context:

- natural logarithm (loge or ln) in mathematical analysis
- common logarithm $\left(\log _{10}\right)$ in engineering and when logarithm tables are used to simplify hand calculations
- binary logarithm $\left(\log _{2}\right)$ in information theory and musical intervals

Here we review some of the rules for logarithmic operations: let $b$ be the base $(b>0$ and $b \neq 1$ ), $c$ be a constant and $x, y, z$ be positive real variables.

Rule 1: (Product Rule)
$\log _{b}(x \cdot y)=\log _{b}(x)+\log _{b}(y) \quad$ Log of a product is the sum of two logs.
Rule 2: (Quotient Rule)
$\log _{b}\left(\frac{x}{y}\right)=\log _{b}(x)-\log _{b}(y) \quad$ Log of a ratio is the difference of two logs.

## Rule 3: (Power Rule)

$\log _{b}\left(x^{c}\right)=c \times \log _{b}(x)$

Rule 4: $\log _{b} b=1$

Rule 5: $\log _{b} 1=0$

Rule 6: (Change of Base)

$$
\log _{b}(x)=\log _{b}(c) \cdot \log _{c}(x)
$$

Since $b^{1}=b$, this rule follows from the definition of logarithm in equation (1.15).

Since $b^{0}=1$, this rule follows from the definition of logarithm in equation (1.15).

For variable $x$, this rule changes the base of logarithm from $b$ to $c$.

Rule 7: (Inverse of Base)
$\log _{b}(c)=\frac{1}{\log _{c}(b)}$
This rule allows one to swap base and argument of a logarithm.

The first five rules are self-explanatory and easy to apply. Rules 6 and 7, on the other hand, may become handy when one wants to compute the value of a logarithm sung a nonengineering type of calculator. Most of the financial calculators only have the natural $\log$ and/or
common $\log$, it would be a problem if you need to calculate a $\log$ with a base different than $e$. For example, how do you want to find the value of $\log _{1.1}(3)$ if only natural $\log$ is available on your calculator?

You can solve this problem by applying both Rule 6 and Rule 7 above:

$$
\begin{array}{rlrl}
\log _{1.1}(3) & =\log _{1.1}(e) \times \ln (3) & & \text { (by Change of Base Rule) } \\
& =\frac{1}{\ln (1.1)} \times \ln (3) & & \\
& =11.5267 & \text { by Inverse of Base Rule) }
\end{array}
$$

## Chapter 1 Homework Problems

Obtain the values of the following:

1. $S=1+2+3+4+\ldots . .+100$
2. $S=1^{2}+2^{3}+2^{4}+2^{5}+\ldots .+2^{100}$
3. $S=1+2+2^{2}+2^{3}+2^{4}+\ldots$. up to $\infty$
4. $S=10+\frac{10}{1+.06}+\frac{10}{(1+.06)^{2}}+\ldots . .+\frac{10}{(1+.06)^{30}}$
5. $S=\frac{d_{1}}{1+k}+\frac{d_{1}(1+g)}{(1+k)^{2}}+\frac{d_{1}(1+g)^{2}}{(1+k)^{3}}+\ldots \ldots+\frac{d_{1}(1+g)^{n-1}}{(1+k)^{n}}$
6. $S=\frac{d_{1}}{1+k}+\frac{d_{1}(1+g)}{(1+k)^{2}}+\frac{d_{1}(1+g)^{2}}{(1+k)^{3}}+\ldots \ldots$ up to $\infty$
7. $S=\frac{10}{1+.06}+\frac{10}{(1+.06)^{2}}+\ldots .+\frac{10}{(1+.06)^{100}}+\frac{10}{(1+.06)^{100}(1+.10)}+\frac{10}{(1+.06)^{100}(1+.10)^{2}}+$

$$
\ldots .+\frac{10}{(1+.06)^{100}(1.10)^{50}}
$$

8. $S=\frac{90}{1.12}+\frac{90}{(1.12)^{2}}+\ldots+\frac{90}{(1.12)^{50}}+\frac{1000}{(1.12)^{50}}$
9. $S=\frac{10}{1+.06}+\frac{10}{(1+.06)^{2}}+\ldots+\frac{10}{(1+.06)^{100}}+\frac{10}{(1+.06)^{100}(1+.10)}+\frac{10}{(1+.06)^{100}(1+.10)^{2}}+\ldots .$. up to $\infty$
10. $S=\frac{10}{(1.10)^{2}}+\frac{10}{(1.10)^{4}}+\frac{10}{(1.10)^{6}}+\frac{10}{(1.10)^{8}}+\ldots . .+\frac{10}{(1.10)^{1000}}$

## Solutions to Chapter 1 Problems

1. $S=1+2+3+4+\ldots . .+100$

In algebra this kind of problem is known as the sum of the first $n$ numbers. The formula for this problem is

$$
S=\frac{n(n+1)}{2}
$$

Where $n=$ number of numbers. In this case, $n=100$. Therefore,

$$
S=\frac{100(100+1)}{2}=50 \times 101=5,050 .
$$

2. $S=1+2^{2}+2^{3}+2^{4}+2^{5}+\ldots .+2^{100}$

Note that in the present form it is not a geometric series because

$$
\frac{\text { Second Term }}{\text { First Term }}=\frac{2^{2}}{1}=4
$$

However,

$$
\frac{\text { Third Term }}{\text { Second Term }}=\frac{\text { Fourth Term }}{\text { Third Term }}=\cdots=2
$$

Therefore, the series is a geometric series only from

$$
2^{2}+2^{3}+2^{4}+2^{5}+\ldots .+2^{100}
$$

Here, First Term $=2^{2}=a$, common ratio $r$ is 2 and the number of terms $n$ is 99 (count repeatedly). Thus, the sum of the geometric series is

$$
\begin{equation*}
\frac{a\left(1-r^{n}\right)}{1-r}=\frac{2^{2}\left(1-2^{99}\right)}{1-2} \tag{S1.1}
\end{equation*}
$$

Therefore, the answer will be

$$
S=1+\frac{2^{2}\left(1-2^{99}\right)}{1-2}
$$

Remember, 1 in

$$
S=1+2^{2}+2^{3}+2^{4}+2^{5}+\ldots .+2^{100}
$$

was not part of the geometric series. Solution (S1.1) is the sum of only

$$
2^{2}+2^{3}+2^{4}+2^{5}+\ldots .+2^{100}
$$

therefore you have to add 1 to arrive the correct answer.
3. $S=1+2+2^{2}+2^{3}+2^{4}+\ldots$ up to $\infty$

By careful examination we note that
(a) This is an infinite geometric series.
(b) The common ratio 2 , which is more than 1 .

Since we know that if in infinite geometric series its common ratio is equal to or greater than 1 , the sum is always $\infty$. Hence, the answer is infinity.
4. $S=10+\frac{10}{1+.06}+\frac{10}{(1+.06)^{2}}+\ldots . .+\frac{10}{(1+.06)^{30}}$

Note that
(a) This is a finite series.
(b) The ratio of any succeeding term over preceding term is constant. That is

$$
r=\frac{\frac{10}{1.06}}{10}=\frac{\frac{10}{(1.06)^{2}}}{\frac{10}{1.06}}=\cdots=\frac{1}{1.06}
$$

(c) Number of terms $n=31$.
(d) First term $a=10$.

Hence the answer will be

$$
\begin{aligned}
S & =\frac{10\left[1-\left(\frac{1}{1.06}\right)^{31}\right]}{1-\frac{1}{1.06}} \\
& =\frac{10\left[1-(1.06)^{-31}\right]}{\frac{1.06-1}{1.06}} \\
& =10\left[\frac{1-(1.06)^{-31}}{.06}\right] \cdot(1.06) \\
& =147.6483
\end{aligned}
$$

5. $S=\frac{d_{1}}{1+k}+\frac{d_{1}(1+g)}{(1+k)^{2}}+\frac{d_{1}(1+g)^{2}}{(1+k)^{3}}+\ldots \ldots+\frac{d_{1}(1+g)^{n-1}}{(1+k)^{n}}$

Note that
(a) This is a finite series.
(b) The ratio of any succeeding term over preceding term is constant. That is

$$
r=\frac{\frac{d_{1}(1+g)}{(1+k)^{2}}}{\frac{d_{1}}{(1+k)}}=\frac{\frac{d_{1}(1+g)^{2}}{(1+k)^{3}}}{\frac{d_{1}(1+g)}{(1+k)^{2}}}=\cdots=\frac{1+g}{1+k}
$$

(c) Number of terms is $n$.
(d) First term is $a=\frac{d_{1}}{1+k}$.

Hence the sum of the geometric series is

$$
\begin{aligned}
S & =\frac{\frac{d_{1}}{(1+k)}\left[1-\left(\frac{1+g}{1+k}\right)^{n}\right]}{1-\frac{1+g}{1+k}} \\
& =\frac{\frac{d_{1}}{(1+k)}\left[1-\left(\frac{1+g}{1+k}\right)^{n}\right]}{\frac{(1+k)-(1+g)}{(1+k)}} \\
& =\frac{\frac{d_{1}}{(1+k)}\left[1-\left(\frac{1+g}{1+k}\right)^{n}\right]}{\frac{k-g}{(1+k)}} \\
& =\frac{d_{1}\left[1-\left(\frac{1+g}{1+k}\right)^{n}\right]}{k-g}
\end{aligned}
$$

6. $S=\frac{d_{1}}{1+k}+\frac{d_{1}(1+g)}{(1+k)^{2}}+\frac{d_{1}(1+g)^{2}}{(1+k)^{3}}+\ldots \ldots$ up to $\infty$

This problem is exactly like problem number 5 except here we are dealing with an infinite geometric series. The sum of the geometric series is

$$
\begin{aligned}
S & =\frac{a}{1-r} & & \text { for } r<1 \\
& =\infty & & \text { for } r \geq 1
\end{aligned}
$$

Therefore, for our problem

$$
\begin{aligned}
S & =\frac{d_{1}}{1+k}\left[1+\frac{1+g}{1+k}+\left(\frac{1+g}{1+k}\right)^{2}+\left(\frac{1+g}{1+k}\right)^{3}+\cdots\right] \\
& =\frac{d_{1}}{1+k}\left[\frac{1}{1-\frac{1+g}{1+k}}\right]=\frac{d_{1}}{1+k} \cdot \frac{1}{\frac{(1+k)-(1+g)}{1+k}} \\
& =\frac{d_{1}}{1+k} \text { for } k>g .
\end{aligned}
$$

Caution: We know that the common ratio is $\frac{1+g}{1+k}$, and we also know that if $\mathrm{CR}=1$ or $\mathrm{CR}>$ 1 , then $S=\infty$.

For $\mathrm{CR}=\frac{1+g}{1+k}$ to be equal to $1, g$ must be equal to $k$; for $\mathrm{CR}=\frac{1+g}{1+k}$ to be greater than $1, g$ must be greater than $k$. So if $g=k$ or $g>k$, we will have $S=\infty$; and for $g<k$ (or $k>g$ ),

$$
S=\frac{d_{1}}{k-g}
$$

7. $S=\frac{10}{1+.06}+\frac{10}{(1+.06)^{2}}+\ldots .+\frac{10}{(1+.06)^{100}}+\frac{10}{(1+.06)^{100}(1+.10)}+\frac{10}{(1+.06)^{100}(1+.10)^{2}}+$ $\ldots .+\frac{10}{(1+.06)^{100}(1.10)^{50}}$

This problem looks sinister, but in reality is not so. Just rewrite the problem as follows,

$$
\begin{align*}
S & =\frac{10}{1+.06}+\frac{10}{(1+.06)^{2}}+\ldots .+\frac{10}{(1+.06)^{100}} \\
& +\frac{10}{(1+.06)^{100}}\left[\frac{1}{1.10}+\frac{1}{(1.10)^{2}}+\frac{1}{(1.10)^{3}}+\cdots+\frac{1}{(1.10)^{50}}\right] \tag{S1.2}
\end{align*}
$$

Note that $S$ is the sum of two geometric series. The first part (the top half of equation (S1.2)) has common ratio of $\frac{1}{1.06}$ while the second part (the bottom half of equation (S1.2)) has common ration of $\frac{1}{1.10}$.

For the first part, the sum is

$$
\begin{equation*}
\frac{\frac{10}{1.06}\left[1-\left(\frac{1}{1.06}\right)^{100}\right]}{1-\frac{1}{1.06}}=10\left[\frac{1-(1.06)^{-100}}{0.06}\right] \tag{S1.3}
\end{equation*}
$$

For the second part, the sum is

$$
\begin{equation*}
\frac{10}{(1.06)^{100}}\left[\frac{\frac{1}{1.10}\left(1-\left(\frac{1}{1.10}\right)^{50}\right)}{1-\frac{1}{1.10}}\right]=\frac{10}{(1.06)^{100}}\left[\frac{1-(1.10)^{-50}}{.10}\right] \tag{S1.4}
\end{equation*}
$$

The answer will be the sum of these two parts, that is, the sum of equations (S1.3) and (S1.4).
8. $S=\frac{90}{1.12}+\frac{90}{(1.12)^{2}}+\ldots+\frac{90}{(1.12)^{50}}+\frac{1000}{(1.12)^{50}}$

We rewrite $S=S_{1}+\frac{1000}{(1.12)^{50}}$, where $S_{1}=\frac{90}{1.12}+\frac{90}{(1.12)^{2}}+\cdots+\frac{90}{(1.12)^{50}}$ is a geometric series with $a=\frac{90}{1.12}, r=\frac{1}{1.12}$ and $n=50$.

Hence, the answer will be

$$
\begin{aligned}
S & =\frac{\frac{90}{1.12}\left[1-\left(\frac{1}{1.12}\right)^{50}\right]}{1-\frac{1}{1.12}}+\frac{1000}{(1.12)^{50}} \\
& =\frac{90\left[1-(1.12)^{-50}\right]}{0.12}+\frac{1000}{(1.12)^{50}} .
\end{aligned}
$$

9. $S=\frac{10}{1+.06}+\frac{10}{(1+.06)^{2}}+\ldots+\frac{10}{(1+.06)^{100}}+\frac{10}{(1+.06)^{100}(1+.10)}+\frac{10}{(1+.06)^{100}(1+.10)^{2}}+\ldots .$. up to $\infty$

Similar to Problem 7, we rewrite the problem into two parts:

$$
\begin{align*}
S & =\frac{10}{1+.06}+\frac{10}{(1+.06)^{2}}+\ldots .+\frac{10}{(1+.06)^{100}} \\
& +\frac{10}{(1+.06)^{100}}\left[\frac{1}{1.10}+\frac{1}{(1.10)^{2}}+\frac{1}{(1.10)^{3}}+\cdots \text { up to } \infty\right] \tag{S1.5}
\end{align*}
$$

Note that the first part of this problem (the top half of equation (S1.5)) is the same as the first part of equation (S1.2) in Problem 7. Its sum is presented in equation (S1.3). The second part of this problem (the bottom part of equation (S1.5)) is an infinite geometric series with common ratio $\frac{1}{1.10}$. Therefore, the answer will be

$$
\begin{aligned}
S & =\frac{\frac{10}{1.06}\left[1-\left(\frac{1}{1.06}\right)^{100}\right]}{1-\frac{1}{1.06}}+\frac{10}{(1.06)^{100}}\left[\frac{1}{1.10}\left(\frac{1}{1-\frac{1}{1.10}}\right)\right] \\
& =\frac{\frac{10}{1.06}\left[1-(1.06)^{-100}\right]}{\frac{1.06-1}{1.06}}+\frac{10}{(1.06)^{100}}\left[\frac{1}{1.10}\left(\frac{1}{\left.\frac{1.10-1}{1.10}\right)}\right]\right. \\
& =10\left[\frac{1-(1.06)^{-100}}{0.06}\right]+\frac{10}{(1.06)^{100}} \cdot \frac{1}{.10}
\end{aligned}
$$

10. $S=\frac{10}{(1.10)^{2}}+\frac{10}{(1.10)^{4}}+\frac{10}{(1.10)^{6}}+\frac{10}{(1.10)^{8}}+\ldots . .+\frac{10}{(1.10)^{1000}}$

This is a finite geometric series with

$$
\begin{aligned}
& a=\frac{10}{(1.10)^{2}} \\
& r=\mathrm{CR}=\frac{1}{(1.10)^{2}} \\
& n=\text { number of terms }=500
\end{aligned}
$$

Therefore, the answer will be

$$
\begin{aligned}
S & =\frac{\frac{10}{(1.10)^{2}}\left[1-\left(\frac{1}{(1.10)^{2}}\right)^{500}\right]}{1-\frac{1}{(1.10)^{2}}} \\
& =\frac{10\left[1-(1.10)^{-1000}\right]}{(1.10)^{2}-1}
\end{aligned}
$$

